Max-Planck-Institut für Mathematik
Bonn

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Preprint Series 2020 (4)

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MPIM 20-4

FREDHOLM CONDITIONS FOR RESTRICTIONS OF INVARIANT PSEUDODIFFERENTIAL OPERATORS TO ISOTYPICAL COMPONENTS

ALEXANDRE BALDARE, RÉMI CÔME, MATTHIAS LESCH, AND VICTOR NISTOR

ABSTRACT. Let Γ be a finite group acting on a smooth, compact manifold M, let $P \in \psi^m(M; E_0, E_1)$ be a Γ-invariant, classical pseudodifferential operator acting between sections of two vector bundles $E_i \to M$, $i = 0, 1$, and let α be an irreducible representation of the group Γ. Then P induces a map $\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$ between the α-isotypical components of the corresponding Sobolev spaces of sections. We prove that the map $\pi_\alpha(P)$ is Fredholm if, and only if, P is α-*elliptic*, a condition defined in terms of the principal symbol of P and the action of Γ on the vector bundles E_i. The proofs are based on the study of the structure of the algebra of invariant pseudodifferential operators on $E_0 \oplus E_1$. These results generalize those in the abelian case (Baldare, Côme, Lesch, Nistor, to appear in J.O.T.), but the proofs in the general case of a finite group are much more difficult and involve new ideas. The result is not true for non-discrete groups.

CONTENTS

M.L. was partially supported by the Hausdorff Center for Mathematics, Bonn.
A.B., R.C., and V.N. have been partially supported by ANR-14-CE25-0012-01 (SINGSTAR). Manuscripts available from **http://www.math.psu.edu/nistor/.**

1. Introduction

Fredholm operators have been extensively studied and appear in many questions in Mathematical Physics, in Partial Differential Equations (linear and non-linear), in Geometry, in Index Theory, and in other areas. On a compact manifold, a classical pseudodifferential operator is Fredholm between suitable Sobolev spaces if, and only if, it is elliptic. In this paper, we obtain an analogous result for the restrictions to isotypical components of a classical pseudodifferential operator P invariant with respect to the action of a finite group Γ. Namely, *the restriction of P to the isotypical component corresponding to an irreducible representation α of Γ is Fredholm if, and only if, the operator is α-elliptic (Theorem 1.2).*

Let us now formulate and explain this result in more detail.

1.1. **The setting and general notation.** We shall work essentially in the same setting as the one considered in [5], but for a general finite group Γ. Thus, *throughout this paper, Γ will be a finite group acting by diffeomorphisms on a smooth Riemannian manifold M.* As our main result is only valid for a compact manifold, we assume for this introduction that M is compact. Certain intermediate results hold, however, also for open manifolds and non-discrete groups, see [5]. For the main result (Theorem 1.2), we do need Γ to be discrete and finite, so our main result is optimal. Again, see [5]. There is no loss of generality to assume that M is endowed with an invariant Riemannian metric, so we will assume that this is the case also.

As usual, $\widehat{\Gamma}$ denotes the finite set of equivalence classes of irreducible Γ-modules (or representations). Let $T : V_0 \to V_1$ be a Γ-equivariant linear map of Γ-modules and $\alpha \in \widehat{\Gamma}$. Then T induces by restriction a Γ-equivariant linear map

$$\pi_\alpha(T) : V_{0\alpha} \to V_{1\alpha} \tag{1}$$

between the α-isotypical components of the Γ-modules V_i, $i = 0, 1$.

We are mostly interested in this restriction morphism π_α in the following case. Let $P \in \psi^m(M; E_0, E_1)$ be a classical, Γ-invariant pseudodifferential operator acting between sections of two Γ-equivariant vector bundles $E_i \to M$, $i = 0, 1$. Then we obtain the operator

$$\pi_\alpha(P) \,:\, H^s(M; E_0)_\alpha \,\to\, H^{s-m}(M; E_1)_\alpha \,, \tag{2}$$

which acts between the α-isotypical components of the corresponding Sobolev spaces of sections. Our main result concerns this operator $\pi_\alpha(P)$. For simplicity, we will consider only classical pseudodifferential operators in this article [44, 46].

1.2. **The α-principal symbol and α-ellipticity.** To put our result into the right perspective, recall that a classical, order m, pseudodifferential operator P is called *elliptic* if its principal symbol

$$\sigma_m(P) \in \mathcal{C}^\infty(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E_0, E_1)) \,, \tag{3}$$

is invertible. Also, recall that a linear operator $T : X_0 \to X_1$ acting between Banach spaces is *Fredholm* if, and only if, the vector spaces

$$\ker(T) \,:=\, T^{-1}(0) \quad \text{and} \quad \mathrm{coker}(T) \,:=\, X_1/TX_0$$

are (both) finite dimensional. Since M is compact, a very well known and widely used result states that $P : H^s(M; E_0) \to H^{s-m}(M; E_1)$ is Fredholm if, and only if, P is elliptic [22, 23, 32, 33, 41]. Consequently, if P is elliptic, then $\pi_\alpha(P)$ is also Fredholm. The converse is not true, however, in general.

To state our main result characterizing the Fredholm property of $\pi_\alpha(P)$ in terms of the "α-principal symbol" $\sigma_m^\alpha(P)$ of P, Theorem 1.2, we shall need to introduce $\sigma_m^\Gamma(P)$, the "Γ-equivariant principal symbol" of P, which is a refinement of the principal symbol $\sigma_m(P)$ of P that takes into account the action of the group Γ. The *α-principal symbol* $\sigma_m^\alpha(P)$ of P is a suitable restriction of the Γ-equivariant principal symbol $\sigma_m^\Gamma(P)$. Let us formulate now the precise definition of these concepts.

The *main question* that we answer in this paper is to determine when the induced operator $\pi_\alpha(P)$ of Equation (2) is Fredholm in terms of its Γ-equivariant principal symbol $\sigma_m^\Gamma(P)$, see Theorem 1.2 below for the precise statement.

The Γ-invariance of P implies that its principal symbol is also Γ invariant:

$$\sigma_m(P) \in \mathcal{C}^\infty(T^*M \smallsetminus \{0\}; \operatorname{Hom}(E_0, E_1))^\Gamma .$$

Let $\Gamma_\xi := \{\gamma \in \Gamma \mid \gamma\xi = \xi\}$ denote the isotropy of a $\xi \in T^*_x M$, $x \in M$, as usual. The isotropy Γ_x of $x \in M$ is defined similarly. Then $\Gamma_\xi \subset \Gamma_x$ acts on E_{0x} and on E_{1x}, the fibers of $E_0, E_1 \to M$ at x. If $Q \in \mathcal{C}^\infty(T^*M \smallsetminus \{0\}; \operatorname{Hom}(E_0, E_1))^\Gamma$, then $Q(\xi) \in \operatorname{Hom}(E_{0x}, E_{1x})^{\Gamma_\xi}$. Let $\rho \in \widehat{\Gamma}_\xi$ be an irreducible representation of Γ_ξ, then

$$\widehat{Q}(\xi, \rho) := \pi_\rho\big[Q(\xi)\big] \in \operatorname{Hom}(E_{0x\rho}, E_{1x\rho})^{\Gamma_\xi} \tag{4}$$

denotes the restriction of Q to the isotypical component corresponding to ρ, with π_ρ defined in Equation (1). Let

$$X_{M,\Gamma} := \{(\xi, \rho) \mid \xi \in T^*M \smallsetminus \{0\} \text{ and } \rho \in \widehat{\Gamma}_\xi\} . \tag{5}$$

Thus Q defines a function on $X_{M,\Gamma}$. Applying this construction to $\sigma_m(P) \in \mathcal{C}^\infty(T^*M \smallsetminus \{0\}; \operatorname{Hom}(E_0, E_1))^\Gamma$ we obtain a function

$$\begin{aligned} \sigma_m^\Gamma(P) : X_{M,\Gamma} \to \bigcup_{(x,\rho)\in X_{M,\Gamma}} \operatorname{Hom}(E_{0x\rho}, E_{1x\rho})^{\Gamma_\xi} , \\ \sigma_m^\Gamma(P)(\xi, \rho) := \pi_\rho(\sigma_m(P)(\xi)) \in \operatorname{Hom}(E_{0x\rho}, E_{1x\rho})^{\Gamma_\xi} , \quad \xi \in T^*_x M . \end{aligned} \tag{6}$$

That is $\sigma_m^\Gamma(P) := \widehat{\sigma_m(P)}$.

The α-principal symbol $\sigma_m^\alpha(P)$ of P, $\alpha \in \widehat{\Gamma}$, is defined in terms of $\sigma_m^\Gamma(P)$, but we need a crucial additional ingredient that takes α into account. The characterization of Fredholm operators can be reduced to each component of the manifold. We shall therefore often assume in this paper that the manifold is connected. This simplifies also the statements and the proofs, so, for the rest of this introduction, we shall assume that our manifold M is connected.

Let A and B be finite groups and let H a subgroup of both A and B. Let $\alpha \in \hat{A}$ and $\beta \in \hat{B}$. We say that α and β are *H-disjoint* if $\operatorname{Hom}_H(\alpha, \beta) = 0$, otherwise we say that they are *H-associated*.

Recall that $\Gamma_{g\xi} = g\Gamma_\xi g^{-1}$ and that this defines an action of Γ on the set $\{\Gamma_\xi \mid \xi \in T^*M\}$ given by $g \cdot \Gamma_\xi = \Gamma_{g\xi}$. For $\rho \in \widehat{\Gamma}_\xi$ define $g \cdot \rho \in \widehat{\Gamma}_{g\xi}$ by $(g \cdot \rho)(h) = \rho(g^{-1}hg)$, for all $h \in \Gamma_{g\xi}$. Let Γ_0 be a minimal isotropy group for M (see Subsection 2.4.3). Let

$$X^\alpha_{M,\Gamma} := \{(\zeta, \rho) \in X_{M,\Gamma} \mid \exists g \in \Gamma,\ g \cdot \rho \text{ and } \alpha \text{ are } \Gamma_0\text{-associated}\}, \tag{7}$$

Thus $(\zeta, \rho) \in X^\alpha_{M,\Gamma}$ if there is a $g \in \Gamma$ such that $g \cdot \rho$ and α are not Γ_0-disjoint.

Definition 1.1. The *α-principal symbol* $\sigma_m^\alpha(P)$ of P is the restriction of $\sigma_m^\Gamma(P)$ to $X_{M,\Gamma}^\alpha$:

$$\sigma_m^\alpha(P) := \sigma_m^\Gamma(P)|_{X_{M,\Gamma}^\alpha}.$$

We shall say that $P \in \psi^m(M; E_0, E_1)^\Gamma$ is *α-elliptic* if its α-principal symbol $\sigma_m^\alpha(P)$ is invertible everywhere on its domain of definition.

1.3. **Statement of the main result.** An alternative formulation of Definition 1.1 is that P is α-elliptic if, and only if, σ_m^Γ is invertible on $X_{M,\Gamma}^\alpha$ (this is, of course, a condition only for those ρ such that $E_{i\rho} \neq 0$, because, otherwise, we get an operator acting on the zero spaces, which we admit to be invertible). We then have the following result extending the classical result (i.e. $\Gamma = \{1\}$) and the one from [5] (i.e. Γ finite abelian) to a general finite group Γ.

Theorem 1.2. *Let Γ be a finite group acting on a smooth, compact manifold M and let $P \in \psi^m(M; E_0, E_1)^\Gamma$ be a Γ-invariant classical pseudodifferential operator acting between sections of two Γ-equivariant bundles $E_i \to M$, $i = 0, 1$, $m \in \mathbb{R}$, and $\alpha \in \widehat{\Gamma}$. We have that*

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$$

is Fredholm if, and only if it is α-elliptic.

As in the abelian case, if Γ acts without fixed points on a *dense* open subset of M, then $X_{M,\Gamma} = X_{M,\Gamma}^\alpha$ for all $\alpha \in \widehat{\Gamma}$, by Corollary 3.19. Hence, in this case, P is α-elliptic if, and only if, it is elliptic. The ellipticity of P can thus be checked in this case simply by looking at the action of P on a single isotypical component. We stress, however, that if Γ is not discrete, this statement, as well as the statement of the above theorem, are no longer true. However, many intermediate results remain valid for compact Lie groups.

A motivation for our result comes from index theory. Let us assume that P is Γ-invariant and elliptic. Atiyah and Singer have determined, for any $\gamma \in \Gamma$, the value at γ of the character of $\mathrm{ind}_\Gamma(P) \in R(G)$. More precisely, they have computed $\mathrm{ind}_\Gamma(P)(\gamma) \in \mathbb{C}$ in terms of data at the fixed points of γ on M [3]. (Here $R(G) := \mathbb{Z}^{\widehat{G}}$ is the representation ring of G and is identified with a subalgebra of $\mathcal{C}^\infty(G)^G$, the ring of conjugacy invariant functions on G via the characters of representations.) By contrast, the multiplicity of $\alpha \in \widehat{\Gamma}$ in $\mathrm{ind}_\Gamma(P)$ was much less studied. It did appear, however, implicitly in the work of Brüning [8, 9], who initiated the program of studying the "isotypical heat trace" $\mathrm{tr}(p_\alpha e^{-t\Delta})$ and its short time asymptotic expansion. Its heat trace is nothing but the heat trace of $\pi_\alpha(\Delta)$. Brüning's program would lead, in particular, to a heat equation determination of the α-isotypical component of the Γ-equivariant index $\mathrm{ind}_\Gamma(D)$ for Dirac type operators D. Carrying out this program is one of the motivations of this paper.

The formulation of our main result does not use C^*-algebras, but its proof does. C^*-algebras were used recently to obtain Fredholm conditions in [15, 17, 28, 37], for example. Some of the algebras involved were groupoid algebras [12, 18, 34, 40]. Fredholm conditions play an important role in the study of the essential spectrum of Quantum Hamiltonians [6, 21, 20, 25, 27]. The technique of "limit operators" [26, 30, 31, 39] is related to groupoids. Some of the most recent papers using related ideas include [4, 11, 12, 13, 35, 36, 47], to which we refer for further references. Besides C^*-algebras, pseudodifferential operators were also used to obtain Fredholm conditions, see [16, 29, 24] and the references therein.

1.4. **Contents of the paper.** We start in Section 2 with some preliminaries. We recall some facts about group actions, most notably the induction of representations and Frobenius reciprocity for finite groups. We also review some notions concerning the primitive spectrum of a C^*-algebra, as well as basic facts concerning (equivariant) pseudodifferential operators.

As in [5], we may assume $E_0 = E_1 = E$ and P to be of order zero. Let $A_M := \mathcal{C}(S^*M; \mathrm{End}(E))$. The most substantial technical results are in Section 3. There, we identify the primitive spectrum of the C^*-algebra A_M^Γ of Γ-invariant symbols with the set $X_{M,\Gamma}/\Gamma$ described above. Some care is taken to describe the corresponding topology on $X_{M,\Gamma}/\Gamma$. We then consider the canonical map from A_M^Γ to the Calkin algebra of $L^2(M; E)_\alpha$ and show that the closed subset of $\mathrm{Prim}(A_M^\Gamma)$ associated to its kernel is $X_{M,\Gamma}^\alpha/\Gamma$.

These descriptions are used in Section 4 to prove the main result of the paper, Theorem 1.2. This section also addresses some particular cases of the Theorem and gives a few examples. We also explain the relation with previously known results, namely:

- the particular formulation in the abelian case, which was established in [5],
- Fredholm conditions for transversally elliptic operators when the group Γ is not discrete,
- Simonenko's local principle for Fredholm operators.

The last named author thanks Max Planck Institute for support while this research was performed.

2. Preliminaries

This section is devoted to background material. For the most part, it will consist of a brief review of sections 2 and 3 of [5], where the reader will find more details, as well as definitions and results not repeated here. Note, however, that we need certain preliminary results for the case Γ non-commutative that were not needed in the abelian case. Nevertheless, the reader familiar with [5] can skip this section at a first reading.

For simplicity, let us assume throughout this paper also that M is connected, the general case being easily reduced to this one since the Fredholm property is valid globally if, and only if, it is valid on each connected component of our compact manifold.

2.1. **Group representations.** We follow the standard terminology and conventions. See, for instance, [5, 7, 42], where one can find further details. Most of the needed basic background material was recalled in greater detail in [5].

Throughout the paper, we denote by Γ a *finite* group acting by isometries on a smooth, Riemannian manifold M (without boundary). We use the standard notations, see [5, 7, 42], to which we refer for further details. If $x \in M$, then Γx is the Γ orbit of x and

$$\Gamma_x := \{\gamma \in \Gamma \mid gx = x\} \subset \Gamma \tag{8}$$

the isotropy group of the action at x.

We shall write $H \sim H'$ if the subgroups H and H' are conjugated in Γ. If $H \subset \Gamma$ is a subgroup, then $M_{(H)}$ will denote the set of elements of M whose isotropy Γ_x is conjugated to H (in Γ), that is, the set of elements $x \in M$ such that $\Gamma_x \sim H$.

Assuming that Γ acts on a space X, we denote by $\Gamma \times_H X$ the space

$$\Gamma \times_H X \,:=\, (\Gamma \times X)/\sim, \tag{9}$$

where $(\gamma h, x) \sim (\gamma, hx)$, $\forall \gamma \in \Gamma, h \in H$ and $x \in X$.

Let V be a normed complex vector space and $\mathcal{L}(V)$ be the set of bounded operators on V. A representation of Γ on V is a group morphism $\Gamma \to \mathcal{L}(V)$; in that case we also call V a Γ-module.

For any two Γ-modules $\mathcal{H}$ and $\mathcal{H}_1$, we shall denote by

$$\mathrm{Hom}_\Gamma(\mathcal{H}, \mathcal{H}_1) \,=\, \mathrm{Hom}(\mathcal{H}, \mathcal{H}_1)^\Gamma \,=\, \mathcal{L}(\mathcal{H}, \mathcal{H}_1)^\Gamma$$

the set of continuous linear maps $T : \mathcal{H} \to \mathcal{H}_1$ that commute with the action of Γ, that is, $T(\gamma\xi) = \gamma T(\xi)$ for all $\xi \in \mathcal{H}$ and $\gamma \in \Gamma$.

Let $\mathcal{H}$ be a Γ-module and α an irreducible Γ-module. Then p_α will denote the Γ-invariant projection onto the α-isotypical component $\mathcal{H}_\alpha$ of

$\mathcal{H}$, defined as the largest (closed) Γ submodule of $\mathcal{H}$ that is isomorphic to a multiple of α. In other words, $\mathcal{H}_\alpha$ is the sum of all Γ-submodules of $\mathcal{H}$ that are isomorphic to α. Notice that $\mathcal{H}_\alpha \simeq \alpha \otimes \mathrm{Hom}_\Gamma(\alpha, \mathcal{H})$.

Since Γ is finite, it is, in particular, compact, and hence we have

$$\mathcal{H}_\alpha \neq 0 \;\Leftrightarrow\; \mathrm{Hom}_\Gamma(\alpha, \mathcal{H}) \neq 0 \;\Leftrightarrow\; \mathrm{Hom}_\Gamma(\mathcal{H}, \alpha) \neq 0. \tag{10}$$

If $T \in \mathcal{L}(\mathcal{H})^\Gamma$ (i.e. T is Γ-equivariant), then $T(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ and we let

$$\pi_\alpha : \mathcal{L}(\mathcal{H})^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha)\,, \quad \pi_\alpha(T) := T|_{\mathcal{H}_\alpha}\,, \tag{11}$$

be the associated morphism, as in Equation (1) of the Introduction. The morphism π_α will play an essential role in what follows.

2.2. **Induction and Frobenius reciprocity.** We now recall some definitions and results for induced representations mainly to set up notation and to obtain some intermediate results.

We let $V^{(I)} := \{f : I \to V\}$ for I finite. If $H \subset \Gamma$ is a subgroup (hence also finite) and V is a H-module, we define, as usual,

$$\begin{aligned} \mathrm{Ind}_H^\Gamma(V) &:= \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} V \\ &\simeq \{\, \xi : \Gamma \to V \mid f(gh^{-1}) = hf(g) \,\} \simeq V^{(\Gamma/H)} \end{aligned} \tag{12}$$

to be the *induced representation*. The last isomorphism is obtained using a set of representatives of the right cosets Γ/H. The action of the group Γ on $\mathrm{Ind}_H^\Gamma(V)$ is obtained from the left multiplication on $\mathbb{C}[\Gamma]$ and the first isomorphism defines the Γ-module structure on $\mathrm{Ind}_H^\Gamma(V)$. The induction is a functor, that is, the Γ-module $\mathrm{Ind}_H^\Gamma(V)$ depends functorially on V.

Remark 2.1. Summarizing Remark 2.2 of [5], we have that

(1) if V is a H-algebra, then $\mathrm{Ind}_H^\Gamma(V)$ is an algebra for the pointwise product,
(2) if V is a left R-module (with compatible actions of Γ), then $\mathrm{Ind}_H^\Gamma(V)$ is a $\mathrm{Ind}_H^\Gamma(R)$ module, again with the pointwise multiplication,
(3) the induction is compatible with morphisms of modules and algebras (change of scalars), again by the function representation of the induced representation.

See [5, Remark 2.2] for more details.

We shall use the *Frobenius reciprocity* in the form that states that we have an isomorphism

(13)
$$\Phi = \Phi^{\Gamma,\mathcal{H}}_{H,V} : \mathrm{Hom}_H(\mathcal{H}, V) \to \mathrm{Hom}_\Gamma(\mathcal{H}, \mathrm{Ind}^\Gamma_H(V)),$$
$$\Phi(f)(\xi) := \frac{1}{|H|}\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} f(g^{-1}\xi), \quad \xi \in \mathcal{H},\ f \in \mathrm{Hom}_H(\mathcal{H}, V).$$

The version of the Frobenius reciprocity used in this paper is valid only for finite groups [7, 42] (although it can be suitably be generalized to the compact case). We note that a more precise notation would be to write $\mathrm{Hom}_H(\mathrm{Res}^\Gamma_{\mathrm{H}}(\mathcal{H}), V)$ instead of our simplified notation $\mathrm{Hom}_H(\mathcal{H}, V)$.

Let $\alpha \in \widehat{\Gamma}$, let $H \subset \Gamma$ be a subgroup, and $\beta \in \widehat{H}$. A useful consequence of the Frobenius reciprocity is that the multiplicity of α in $\mathrm{ind}^\Gamma_H(\beta)$ is the same as the multiplicity of β in the restriction of α to H. In particular, α is contained in $\mathrm{ind}^\Gamma_H(\beta)$ if, and only if, β is contained in the restriction of α to H, in which case we say that α and β are H-*associated* (to each other). On the other hand, recall that if β is not contained in the restriction of α to H, we say that α and β are H-disjoint.

Let V be a H-module and $\mathcal{H}$ be the trivial Γ-module $\mathbb{C}$. Then we obtain, in particular, an isomorphism

(14)
$$\Phi : V^H = \mathrm{Hom}_H(\mathbb{C}, V) \simeq \mathrm{Hom}_\Gamma(\mathbb{C}, \mathrm{Ind}^\Gamma_H(V)) = \mathrm{Ind}^\Gamma_H(V)^\Gamma,$$
$$\Phi(\xi) := \frac{1}{|H|}\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} \xi = \sum_{x\in\Gamma/H} x \otimes \xi.$$

If V is an algebra, then the map Φ is an isomorphism of algebras. In particular, we obtain the following consequences.

Remark 2.2. Let $H \subset \Gamma$ be a subgroup of Γ, β_j be non-isomorphic simple H-modules, $j = 1, \ldots, N$, and

(15)
$$\beta := \oplus_{j=1}^N \beta_j^{k_j}.$$

We then have that $\mathrm{Ind}^\Gamma_H(\beta) \simeq \oplus_{j=1}^N \mathrm{Ind}^\Gamma_H(\beta_j^{k_j})$ and the Frobenius isomorphism gives

(16)
$$\mathrm{Ind}^\Gamma_H(\mathrm{End}(\beta))^\Gamma \simeq \mathrm{End}(\beta)^H \simeq \oplus_{j=1}^N \mathrm{End}(\beta_j^{k_j})^H \simeq \oplus_{j=1}^N M_{k_j}(\mathbb{C}),$$

which is a semi-simple algebra and where the first isomorphism is induced by Φ of Equation (14).

We shall need the following refinement of the above remark.

Lemma 2.3. *Let* $\beta := \oplus_{j=1}^{N} \beta_j^{k_j}$ *be as in Equation* (15)*, let*

$$T \,=\, (T_j) \in \mathrm{End}(\beta)^H \,\simeq\, \oplus_{j=1}^{N} \mathrm{End}(\beta_j^{k_j})^H\,,$$

with $T_j \in \mathrm{End}(\beta_j^{k_j})^H$*, and let* $\xi_j \in \mathrm{Ind}_H^\Gamma(\beta_j^{k_j})$*. We let*

$$\xi \,:=\, (\xi_j) \in \oplus_{j=1}^{N} \mathrm{Ind}_H^\Gamma(\beta_j^{k_j}) \simeq \mathrm{Ind}_H^\Gamma(\beta)\,.$$

Then $\Phi(T)(\xi) = (\Phi(T_j)\xi_j)_{j=1,\dots,N}$.

Proof. See for example [5, Lemma 2.4]. □

For the abelian case, the following elementary result was proved in [5] Proposition 2.5. That proof *does not* generalize to our case.

Proposition 2.4. *Let* $\beta := \oplus_{j=1}^{N} \beta_j^{k_j}$ *be as in Equation* (15)*. Let* $J \subset \{1, 2, \dots, N\}$ *be the set of indices* j *such that* α *and* β_j *are* H*-disjoint (i.e.* β_j *is not contained in the restriction of* α *to* H*). Then the morphism*

$$\pi_\alpha : \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \to \mathrm{End}(p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta))$$

is such that

$$\ker(\pi_\alpha) \,=\, \bigoplus_{j\in J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma \text{ and } \mathrm{Im}(\pi_\alpha) \,\simeq\, \bigoplus_{j\notin J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma\,.$$

Proof. By Lemma 2.3, we can assume that $N = 1$. Therefore the algebra $\mathrm{End}(\beta)^H$ is simple (more precisely, isomorphic to a matrix algebra $M_q(\mathbb{C})$, $q = k_1$). We shall use the isomorphism of Equation (16). The action of $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \simeq \mathrm{End}(\beta)^H \simeq M_q(\mathbb{C})$ on $\mathrm{Ind}_H^\Gamma(\beta)$ is unital (i.e. non-degenerate), so the morphism

$$M_q(\mathbb{C}) \,\simeq\, \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \to \mathrm{End}(p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta)) \tag{17}$$

is injective if, and only if, $p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta) \neq 0$. Notice the following equivalences

$$p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta) \neq 0 \Leftrightarrow \mathrm{Hom}(\alpha, \mathrm{Ind}_H^\Gamma(\beta))^\Gamma \neq 0 \Leftrightarrow \mathrm{Hom}(\alpha, \beta)^H \neq 0.$$

The result then follows from Equation (10). □

2.3. **The primitive ideal spectrum of a C^*-algebra.** We shall need a few basic concepts and facts about C^*-algebras. A general reference is [19]. Recall that a two-sided ideal $I \subset A$ of a C^*-algebra A is called *primitive* if it is the kernel of a non-zero, irreducible $*$-representation of A. Hence A is *not* a primitive ideal of itself. By $\mathrm{Prim}(A)$ we shall denote the set of primitive ideals of A, called the *primitive ideal spectrum* of A. If X is a locally compact space, then $\mathcal{C}_0(X)$ denotes the space of continuous functions $X \to \mathbb{C}$ that vanish at infinity. The concept of primitive ideal spectrum is important for us since we have a natural homeomorphism

$$\mathrm{Prim}(\mathcal{C}_0(X)) \simeq X \,. \tag{18}$$

This identification lies at the heart of non-commutative geometry [14].

If A is a type I C^*-algebra, then $\mathrm{Prim}(A)$ identifies with the set of isomorphism classes of irreducible representations of A. Any C^*-algebra with only finite dimensional irreducible representations is a type I algebra [19]. Most of the algebras considered in this paper (a notable exception are the algebras of compact operators), have this property.

The following example from [5] will be used several times.

Example 2.5. Let H be a finite group and $\beta = \oplus_{j=1}^{N} \beta_j^{k_j}$ be as in Remark 2.2. Then, as explained in that remark, $\mathcal{L}(\beta)^H \simeq \oplus_j M_{k_j}(\mathbb{C})$. The algebra $\mathcal{L}(\beta)^H = \mathrm{End}_H(\beta)$ is thus a C^*-algebra with only finite dimensional representations and we have natural homeomorphisms

$$\mathrm{Prim}(\mathrm{End}_H(\beta)) \leftrightarrow \{\beta_1, \beta_2, \ldots, \beta_N\} \leftrightarrow \{1, 2, \ldots, N\} \,.$$

The space $\mathrm{Prim}(A)$ is a topological space for the Jacobson topology: we refer to [19] for more details. We will recall some facts about this topology when we need it, see Lemma 3.2 below.

We shall need the following "central character" map.

Remark 2.6. Let Z be a commutative C^*-algebra and $\phi : Z \to M(A)$ be a $*$-morphism to the multiplier algebra $M(A)$ of A [1, 10]. Assume that $\phi(Z)$ commutes with A and $\phi(Z)A = A$. Then Schur's lemma gives that there exists a natural continuous map

$$\phi^* : \mathrm{Prim}(A) \to \mathrm{Prim}(Z) \,, \tag{19}$$

which we shall call also the *central character map* (associated to ϕ).

We conclude our discussion with the following simple result.

Lemma 2.7. *We freely use the notation of Example 2.5. The inclusion of the unit* $\mathbb{C} \to \mathrm{End}_H(\beta)$ *induces a morphism* $j : \mathcal{C}_0(X) \to \mathcal{C}_0(X; \mathrm{End}_H(\beta)) \simeq \mathcal{C}_0(X) \otimes \mathrm{End}_H(\beta)$. *The resulting central character map is the first projection*

$$j^* : \mathrm{Prim}(\mathcal{C}_0(X; \mathrm{End}_H(\beta))) \simeq X \times \{1, 2, \ldots, N\} \to X \simeq \mathrm{Prim}(\mathcal{C}_0(X)) \,. \tag{20}$$

2.4. **Group actions on manifolds.** As before, we consider a finite group Γ acting by isometries on a compact Riemannian manifold M.

2.4.1. *Slices and tubes.* Given $x \in M$, the isotropy group Γ_x acts linearly and isometrically on T_xM. For $r > 0$, let $U_x := (T_xM)_r$ denote the set of vectors of length $< r$ in N_x. It is known then that, for $r > 0$ small enough, the exponential map gives a Γ-equivariant isometric diffeomorphism

$$W_x = \exp(\Gamma \times_{\Gamma_x} U_x) \simeq \Gamma \times_{\Gamma_x} U_x \tag{21}$$

where W_x is a Γ-invariant neighborhood of x in M and $\Gamma \times_{\Gamma_x} U_x$ is defined in equation (9). More precisely, W_x is the set of $y \in M$ at distance $< r$ to the orbit Γx, if $r > 0$ is small enough. The set W_x is called a *tube* around x (or Γx) and the set U_x is called the *slice* at x. When M is compact, the injectivity radius is bounded from below, so we may assume that the constant r does not depend on x.

2.4.2. *Equivariant vector bundles.* Let us consider now a Γ-equivariant smooth vector bundle $E \to M$. Let us fix $x \in M$ and consider as above the tube $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ around x, see Equation (21). We use this diffeomorphism to identify U_x to a subset of M, in which case, we can also assume the restriction of E to the slice U_x to be trivial. Therefore, there exists a Γ_x-module β such that

$$\begin{aligned} E|_{U_x} \ &\simeq \ U_x \times \beta \ \text{ and} \\ E|_{W_x} \ &\simeq \ \Gamma \times_{\Gamma_x} (U_x \times \beta)\,, \end{aligned} \tag{22}$$

The second isomorphism is Γ-equivariant.

Assume E is endowed with a Γ-invariant hermitian metric. We then have isomorphisms of Γ-modules:

$$\begin{aligned} L^2(W_x; E|_{W_x}) &\simeq \operatorname{Ind}_{\Gamma_x}^{\Gamma}(L^2(U_x;\beta)) \ \text{ and} \\ \mathcal{C}_0(W_x; E|_{W_x}) &\simeq \operatorname{Ind}_{\Gamma_x}^{\Gamma}(\mathcal{C}_0(U_x;\beta))\,. \end{aligned} \tag{23}$$

In view of the previous isomorphism, we will often identify W_x and $\Gamma \times_{\Gamma_x} U_x$, making no distinction between them to simplify notations.

2.4.3. *The principal orbit bundle.* Recall that $M_{(H)}$ denotes the set of points of M whose stabilizer is conjugated in Γ to H. Recall that we have assumed that M is *connected*. We moreover assume that M is compact. The reason of these assumptions is that it is known then [45] that there exists a *minimal isotropy* subgroup $\Gamma_0 \subset \Gamma$, in the sense that $M_{(\Gamma_0)}$ is a dense open subset of M.

In particular, the fact that M is connected gives that there exist minimal elements for the set of isotropy groups of points in M (with respect to inclusion) and all minimal isotropy groups are conjugated to a fixed subgroup $\Gamma_0 \subset \Gamma$. By the definition, the set $M_{(\Gamma_0)}$ consists of the points whose stabilizer is conjugated to that minimal subgroup.

The set $M_{(\Gamma_0)}$ is called the *principal orbit bundle* of M. We will denote $M_{(\Gamma_0)}$ by M_0 in the sequel.

The principal orbit bundle $M_0 := M_{(\Gamma_0)}$ has to following useful property. If $x \in M_0$, then Γ_x acts *trivially* on the slice U_x at x, by the minimality of Γ_0. Hence Γ_0 acts trivially on T_x^*M as well, which implies that $\Gamma_0 \subset \Gamma_\xi$ for any $\xi \in T_x^*M$. If, on the other hand, $x \in M$ is arbitrary (not necessarily in the principal orbit bundle), then the isotropy of Γ_x will contain a subgroup conjugated to Γ_0.

2.5. **Pseudodifferential operators.** We continue to follow [5]. We also continue to assume that Γ is a finite Lie group that acts smoothly and isometrically on a smooth Riemannian manifold M. Let $\psi^m(M;E)$ denote the space of order m, *classical* pseudodifferential operators on M with *compactly supported* distribution kernel.

Let $\overline{\psi^0}(M;E)$ and $\overline{\psi^{-1}}(M;E)$ denote the norm closures of $\psi^0(M;E)$ and $\psi^{-1}(M;E)$, respectively. The action of Γ then extends to an action on $\psi^m(M;E)$, $\overline{\psi^0}(M;E)$, and $\overline{\psi^{-1}}(M;E)$. We shall denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators acting on a Hilbert space $\mathcal{H}$. Of course, we have $\overline{\psi^{-1}}(M;E) = \mathcal{K}(L^2(M;E))$, since we have considered only pseudodifferential operators with compactly supported distribution kernels.

Let S^*M denote the *unit cosphere bundle* of a smooth manifold M, that is, the set of unit vectors in T^*M, as usual. We shall denote, as usual, by $\mathcal{C}_0(S^*M;\mathrm{End}(E))$ the set of continuous sections of the *lift* of the vector bundle $\mathrm{End}(E) \to M$ to S^*M.

Corollary 2.8. *We have an exact sequence*

$$0 \to \mathcal{K}(L^2(M;E))^\Gamma \to \overline{\psi^0}(M;E)^\Gamma \xrightarrow{\sigma_0} \mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma \to 0\,.$$

Proof. See, for instance, [5, Corollary 2.7]. □

2.5.1. *The structure of regularizing operators.* From now on, all our vector bundles will be Γ-equivariant vector bundles. We want to understand the structure of the algebra $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$, for any fixed $\alpha \in \widehat{\Gamma}$ (see Equations (1) and (11) for the definition of the restriction morphism π_α and of the projectors $p_\alpha \in C^*(\Gamma)$).

We shall need the following standard result about negative order operators. Recall that, for $\alpha \in \hat{\Gamma}$, we let π_α be the representation of $\overline{\psi^0}(M;E)^\Gamma$ on $L^2(M;E)_\alpha$ defined by restriction as before, Equations (1) and (11).

Proposition 2.9. *We have the identifications*

$$\begin{aligned} p_\alpha \psi^{-1}(M;E)^\Gamma &\simeq \pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma) \\ &= \pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma) = \mathcal{K}(L^2(M;E)_\alpha)^\Gamma , \end{aligned}$$

where the first isomorphism map is simply π_α *and*

$$\mathcal{K}(L^2(M;E))^\Gamma = \overline{\psi^{-1}}(M;E)^\Gamma \simeq \oplus_{\alpha\in\widehat{\Gamma}} \mathcal{K}(L^2(M;E)_\alpha)^\Gamma .$$

Proof. See, for example, [5, Section 3] for a proof. □

3. The principal symbol

From now on we assume that M is **compact** and connected. Let us fix an irreducible representation α of Γ and consider the restriction morphism π_α to the α-isotypical component of $L^2(M;E)$. Recall that this morphism was first introduced in Equation (1) and discussed in detail in Section 2.1. As in [5], we now turn to the identification of the quotient

$$\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma).$$

The methods used in this paper diverge, however, drastically from those of [5].

Since $\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$ was identified in the previous section, the promised identification of the quotient $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$ will give further insight into the structure of the algebra $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$ and will provide us, eventually, with Fredholm conditions. Recall that, in this paper, we are assuming Γ to be finite. Nevertheless, a several intermediate results hold also in the case Γ compact.

3.1. The primitive ideal spectrum of A_M^Γ.

As before, S^*M denotes the unit cosphere bundle of M. For the simplicity of the notation, we shall write

$$A_M := \mathcal{C}(S^*M; \mathrm{End}(E)),$$

as in the Introduction. Recall from Corollary 2.8 that we have an algebra isomorphism

$$\overline{\psi^0}(M;E)^\Gamma/\overline{\psi^{-1}}(M;E)^\Gamma \simeq A_M^\Gamma. \tag{24}$$

In our case, the inclusion $j : \mathcal{C}(S^*M/\Gamma) \subset A_M^\Gamma$ as a central subalgebra induces, as in Equation (19), a central character map

$$j^* : \mathrm{Prim}(A_M^\Gamma) \to S^*M/\Gamma,$$

that underscores the local nature of the structure of the primitive ideal spectrum of A_M^Γ. We introduce the representation $\pi_{\xi,\rho}$ defined for any $f \in A_M^\Gamma$ by

$$\pi_{\xi,\rho}(f) \,=\, \pi_\rho(f(\xi)),$$

that is $\pi_{\xi,\rho}(f)$ is the restriction of $f(\xi) \in \mathrm{End}(E_x)$ to the ρ-isotypical component of E_x. The central character map j^* was used in [5], Corollary 4.2, to obtain the following identification of $\mathrm{Prim}(A_M^\Gamma)$.

Proposition 3.1 ([5])**.** *Let Ω_M be the set of pairs (ξ, ρ), where $\xi \in S_x^*M$, $x \in M$, and $\rho \in \widehat{\Gamma}_\xi$ appears in E_x (i.e. $\mathrm{Hom}_{\Gamma_\xi}(\rho, E_x) \neq 0$).*

(1) The map $\Omega_M/\Gamma \ni \Gamma(\xi,\rho) \mapsto \ker(\pi_{\xi,\rho}) \in \mathrm{Prim}(A_M^\Gamma)$ is bijective.
(2) The central character map $\Omega_M/\Gamma \simeq \mathrm{Prim}(A_M^\Gamma) \to S^*M/\Gamma$ maps $\Gamma(\xi,\rho) \in \Omega_M/\Gamma$ to $\Gamma\xi$ and is continuous and finite-to-one.

The space $\mathrm{Prim}(A_M^\Gamma)$ is endowed with the Jacobson topology, which was recalled in Subsection 2.3; thus Proposition 3.1 allows us to use the central character map j^* to obtain a topology on Ω_M/Γ that will play a crucial role in what follows. We thus now turn to the study of this topology on Ω_M/Γ. We begin with the following standard lemma.

Lemma 3.2. *Let A be a C^*-algebra. The family $(V_a)_{a\in A}$ defined by*

$$V_a = \{J \in \mathrm{Prim}\, A \mid a \notin J\},$$

for any $a \in A$, is a basis of open sets for $\mathrm{Prim}(A)$.

Proof. Following [19], we know that the open, non-empty subsets of $\mathrm{Prim}(A)$ are exactly the sets

$$\{J \in \mathrm{Prim}(A) \mid I \not\subset J\} \simeq \mathrm{Prim}(I)$$

where I ranges through the closed, non-zero, two-sided ideals of A. If $a \in A$, let us denote by $I_a := \overline{AaA}$ the closed, two-sided ideal generated by a. Then $a \notin J \Leftrightarrow I_a \not\subset J$, and hence $V_a = \mathrm{Prim}(I_a)$. This shows that V_a is open.

Next, let $V \subset \mathrm{Prim}(A)$ be a non-empty open subset and $J_0 \in V$. We know then that there exists a closed, two-sided ideal I, $0 \neq I \subset A$, such that $V = \mathrm{Prim}(I)$. We have $I \not\subset J_0$, and hence we can choose $a \in I \smallsetminus J_0$. If $J \subset A$ is a primitive ideal such that $a \notin J$, then *a fortiori* $I \not\subset J$. Therefore $V_a \subset \mathrm{Prim}(I)$. This shows that $J_0 \in V_a \subset V$. Therefore the family $(V_a)_{a\in A}$ is a basis for the topology on $\mathrm{Prim}(A)$. □

We shall use the bijection of Proposition 3.1 to conclude the following.

Corollary 3.3. *A basis for the induced topology on $\Omega_M/\Gamma \simeq \mathrm{Prim}(A_M^\Gamma)$ is given by the sets*

$$V_f := \{\, \Gamma(\xi,\rho) \in \Omega_M/\Gamma \mid \pi_{\xi,\rho}(f) \neq 0 \,\},$$

where f ranges through the non-zero elements of A_M^Γ.

3.2. **The restriction morphisms.** Let $\mathcal{O} \subset M$ be an open subset. Then $S^*\mathcal{O}$ is the restriction of S^*M to $\mathcal{O}$. We shall need the algebras

$$A_{\mathcal{O}} := \mathcal{C}_0(S^*\mathcal{O}; \mathrm{End}(E)) \quad \text{and} \quad B_{\mathcal{O}} := \overline{\psi^0}(\mathcal{O}; E)\,. \tag{25}$$

Assume that $\mathcal{O} \subset M$ is Γ-invariant. The group Γ does not act, in general, as multipliers on the C^*-algebra $B_{\mathcal{O}} := \overline{\psi^0}(\mathcal{O}; E)$ (it does however act by conjugation), so the method used in [5] to compute $\overline{\psi^{-1}}(\mathcal{O}; E)^\Gamma \simeq \mathcal{K}(L^2(\mathcal{O}; E))^\Gamma$ does not extend to compute $B_{\mathcal{O}}^\Gamma$. We shall thus consider the natural, surjective map

$$\begin{aligned}\mathcal{R}_{\mathcal{O}} : A_{\mathcal{O}}^\Gamma := \mathcal{C}_0(S^*\mathcal{O}; \mathrm{End}(E))^\Gamma &\simeq B_{\mathcal{O}}^\Gamma / \overline{\psi^{-1}}(\mathcal{O}; E)^\Gamma \\ &\to \pi_\alpha(B_{\mathcal{O}}^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(\mathcal{O}; E)^\Gamma).\end{aligned} \tag{26}$$

Recall from Corollary 2.9 that $\pi_\alpha(\overline{\psi^{-1}}(M; E)^\Gamma) = \mathcal{K}(L^2(M; E)_\alpha)^\Gamma$. Therefore, for a given $P \in \overline{\psi^0}(M; E)$, we have that $\pi_\alpha(P)$ is Fredholm if, and only if, the principal symbol of P is invertible in $A_M^\Gamma / \ker(\mathcal{R}_M)$. This will be discussed in more detail in the next section.

We shall approach the computation of $\ker(\mathcal{R}_M) \subset A_M^\Gamma$ by determining the closed subset

$$\Xi := \mathrm{Prim}(A_M^\Gamma / \ker(\mathcal{R}_M)) \subset \mathrm{Prim}(A_M^\Gamma) \tag{27}$$

of the primitive ideal spectrum of A_M^Γ corresponding to $\ker(\mathcal{R}_M)$. Once we will have determined Ξ, we will also have determined $\ker(\mathcal{R}_M)$, in view of the definitions recalled in Subsection 2.3 that put in bijection the closed, two-sided ideals of a C^*-algebra with the closed subsets of its primitive ideal spectrum.

Since $\mathcal{C}(M/\Gamma) \subset B_M$, it follows from the definition of $\mathcal{R}_M$ that it is a $\mathcal{C}(M/\Gamma)$–module morphism, and hence that $\ker(\mathcal{R}_M)$ is a $\mathcal{C}(M/\Gamma)$–module. Let us also recall that

$$\mathcal{C}(M/\Gamma) = \mathcal{C}(M)^\Gamma \subset Z_M := \mathcal{C}(S^*M)^\Gamma \subset Z(A_M^\Gamma) \subset A_M^\Gamma \subset A_M\,.$$

The local nature of $\ker(\mathcal{R}_M)$ and of the space Ξ is explained in the following remark.

Remark 3.4. Let $M/\Gamma = \cup V_k$ be an open cover and $\ker(\mathcal{R}_M)_{V_k} := \mathcal{C}_0(V_k)\ker(\mathcal{R}_M) = \ker(\mathcal{R}_{V_k})$. If we determine each $\ker(\mathcal{R}_M)_{V_k}$, then we determine $\ker(\mathcal{R}_M)$ using a partition of unity through:

$$\ker(\mathcal{R}_M) = \sum_k{}' \phi_k \ker(\mathcal{R}_{V_k})\,, \tag{28}$$

where $\sum'$ refers to sums with only finitely many non-zero terms and (ϕ_k) is a partition of unity of M/Γ with continuous functions subordinated to the covering (V_k) (thus, in particular, $\mathrm{supp}(\phi_k) \subset V_k$). Since

M is compact, we can assume the covering to be finite. (If M was non-compact, then we would need to take the closure of the right hand side in Equation (28).) To determine $\mathcal{R}_M$, we can therefore replace M by any of the open sets V_k in the covering and study $\ker(\mathcal{R}_{V_k})$. We shall do that for the covering of M/Γ with the tubes $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ considered in 2.4.1, see Equation (21).

3.3. **Local calculations.** In view of Remark 3.4, we shall concentrate now on the local structure of $\ker(\mathcal{R}_M)$, that is, on the structure of $\ker(\mathcal{R}_{\mathcal{O}})$ for suitable ("small") open, Γ-invariant subsets $\mathcal{O} \subset M$. Let us fix then $x \in M$ and let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ be the tube around x, Equation (21). For simplicity, we shall write

$$A_x := A_{U_x} := \mathcal{C}_0(S^*U_x; \mathrm{End}(E)) \ \text{ and } \ Z_x := Z(A_x^{\Gamma_x}) . \tag{29}$$

For these algebras, the role of Γ will be played by Γ_x. For the statement of the following lemma, recall the definitions in Subsection (2.4), especially Equation (21).

Lemma 3.5. *Let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$. Then $S^*W_x \simeq \Gamma \times_{\Gamma_x} S^*U_x$ and we have Γ-equivariant algebra isomorphisms*

$$A_{W_x} := \mathcal{C}_0(S^*W_x; \mathrm{End}(E)) \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma} \big(\mathcal{C}_0(S^*U_x; \mathrm{End}(E))\big) =: \mathrm{Ind}_{\Gamma_x}^{\Gamma}(A_x) .$$

Consequently, the Frobenius isomorphism Φ of Equation (14) *induces an isomorphism*

$$\Phi^{-1} : A_{W_x}^{\Gamma} \to A_x^{\Gamma_x} .$$

Proof. We have that $E|_{W_x} \simeq \Gamma \times_{\Gamma_x} (E|_{U_x})$ and hence $\mathrm{End}(E)|_{W_x} \simeq \Gamma \times_{\Gamma_x} (\mathrm{End}(E)|_{U_x})$. Equation (23) then gives that $\mathcal{C}_0(W_x, \mathrm{End}(E)) \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(\mathcal{C}_0(U_x, \mathrm{End}(E)))$. The rest follows right away from the Frobenius reciprocity (more precisely, from Equation (14)) and from Equation (23), with β replaced with $\mathrm{End}(E)$. ☐

Remark 3.6. In view of Equation (14), the isomorphism Φ of Lemma 3.5 can be written explicitly as follows. Let $f \in A_x^{\Gamma_x}$. Then, for any equivalence class $[\gamma, \xi] := \Gamma_x(\gamma, \xi) \in \Gamma \times_{\Gamma_x} S^*U_x \simeq S^*W_x$ we have

$$\Phi(f)([\gamma, \xi]) = [\gamma, f(\xi)],$$

where $[\gamma, f(\xi)] \in \Gamma \times_{\Gamma_x} (U_x \times \mathrm{End}(E_x))^{\Gamma_x} \simeq \Gamma \times_{\Gamma_x} \mathrm{End}(E|_{U_x})^{\Gamma_x} \simeq \mathrm{End}(E|_{W_x})^{\Gamma}$. This defines $\Phi(f) \in \mathcal{C}_0(S^*W_x; \mathrm{End}(E|_{W_x}))^{\Gamma} = A_{W_x}^{\Gamma}$.

Lemma 3.5 together with the following remark will allow us to reduce the study of the algebra A_M^{Γ} to that of its analogues defined for slices.

Remark 3.7. Let U be an open set of some euclidean space and $W = U \times \{1, 2, \ldots, N\}$, where the space on the second factor is endowed with the discrete topology. For simplicity, we identify $L^2(W)$ with $L^2(U)^N$ using the map $f \mapsto (f(i))_{i=1\cdots N}$. Then

$$\begin{gathered}\psi^{-1}(W) = M_N(\psi^{-1}(U)) \simeq \psi^{-1}(U) \otimes M_N(\mathbb{C}) \text{ and hence} \\ \overline{\psi^{-1}}(W) = M_N(\overline{\psi^{-1}}(U)) \simeq \overline{\psi^{-1}}(U) \otimes M_N(\mathbb{C})\,. \end{gathered} \tag{30}$$

On the other hand, if A^N denotes the direct sum of N-copies of the algebra A, then we have the following inclusions of algebras

$$\begin{gathered}\psi^0(U)^N \subset \psi^0(W) \subset M_N(\psi^0(U)) \simeq \psi^0(U) \otimes M_N(\mathbb{C}), \text{ and hence} \\ \overline{\psi^0}(U)^N \subset \overline{\psi^0}(W) \subset M_N(\psi^0(U)) \simeq \overline{\psi^0}(U) \otimes M_N(\mathbb{C})\,. \end{gathered} \tag{31}$$

The following lemma makes explicit the group actions in the isomorphisms of the last remark. Thus, in analogy with the definitions of the algebras $A_{W_x} = \mathcal{C}_0(S^*W_x; \mathrm{End}(E))$ and $A_x = \mathcal{C}_0(S^*U_x; \mathrm{End}(E))$, we consider the algebras

$$B_{W_x} := \overline{\psi^0}(W_x; E) \quad \text{and} \quad B_x := \overline{\psi^0}(U_x; E)\,. \tag{32}$$

We shall also use the standard notation $V^{(I)} := \{f : I \to V\}$ for I finite, as before.

Lemma 3.8. *We keep the notation of Lemma 3.5 and of Equation* (32) *above. Then we have* Γ*-equivariant algebra isomorphisms*

$$B_{W_x} \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(B_x) + \overline{\psi^{-1}}(W_x; E)\,.$$

Consequently, $B_{W_x}^{\Gamma} \simeq \Phi(B_x^{\Gamma_x}) + \overline{\psi^{-1}}(W_x; E)^{\Gamma}$.

Proof. Since $B_y = B_{U_y} \subset B_{W_x}$ for all $y \in \Gamma x$ and since U_x and U_y are diffeomorphic through any $\gamma \in \Gamma$ such that $\gamma x = y$ we obtain the inclusion $B_x^{(\Gamma/\Gamma_x)} \subset B_{W_x}$, as in Remark 3.7. Similarly, since $B_x \to A_x$ is surjective, we obtain the equality $B_{W_x} = B_x^{(\Gamma/\Gamma_x)} + \overline{\psi^{-1}}(W_x; E)$ as in the same remark. From Equation (24) and Lemma 3.5 we know that $B_{W_x}/\overline{\psi^{-1}}(W_x; E) \simeq A_{W_x} \simeq A_{U_x}^{(\Gamma/\Gamma_x)} = \mathrm{Ind}_{\Gamma_x}^{\Gamma}(A_x)$, and hence we obtain $B_{W_x} \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(B_x) + \overline{\psi^{-1}}(W_x; E)$. The last isomorphism follows from the Frobenius reciprocity (more precisely, from Equation (23), with β replaced with B_x) and from the exactness of the functor $V \to V^{\Gamma}$. ☐

To be able to make further progress, it will be convenient to look first at the case when $x \in M$ has minimal isotropy $\Gamma_x \sim \Gamma_0$, that is, when x belongs to the principal orbit bundle $M_0 := M_{(\Gamma_0)}$. The notation Γ_0 will remain fixed from now on.

3.4. **Calculations for the principal orbit bundle.** We assume as before that M is connected. Let Γ_0 be a minimal isotropy group (which, we recall, is unique up to conjugation). Let $x \in M$ be our fixed point and Γ_x its isotropy, as before. The case when Γ_x is conjugated to Γ_0 is simpler since, as noticed already, then Γ_x acts trivially on U_x.

Let us fix $x \in M$ with isotropy group $\Gamma_x = \Gamma_0$. As before, we let

$$W_x \simeq \Gamma \times_{\Gamma_0} U_x \text{ and } E_{|W_x} \simeq \Gamma \times_{\Gamma_0} (U_x \times \beta)\,,$$

where β is some Γ_0-module, as in Equations (21) and (22). We decompose β into a direct sum of representations of the form $\beta_j^{k_j}$ for some non-isomorphic irreducible module (or representation) β_j of Γ_0, again as before:

$$E_x \;=\; \beta \simeq \oplus \beta_j^{k_j}\,.$$

Remark 3.9. We have noticed earlier that Γ_0 acts trivially on U_x, hence on $T^*_x M$. In particular S^*M also has Γ_0 as minimal isotropy subgroup, and S^*M_0 is a dense subset of the principal bundle of S^*M.

Corollary 3.10. *Let $x \in M$ be such that $\Gamma_x = \Gamma_0$ and $\beta = \oplus_{j=1}^N \beta_j^{k_j}$, for some non-isomorphic, irreducible Γ_0-modules β_j. Then*

$$A^{\Gamma}_{W_x} \;\simeq\; A^{\Gamma_x}_x \;\simeq\; \mathcal{C}_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta) \;\simeq\; \oplus_{j=1}^N M_{k_j}\big(\mathcal{C}_0(S^*U_x)\big).$$

In particular, the canonical central character map $\mathrm{Prim}(A_x^{\Gamma_0}) \to S^*U_x \simeq \mathrm{Prim}(\mathcal{C}_0(S^*U_x)^{\Gamma_0})$ *of Proposition 3.1 corresponds to the trivial finite covering* $S^*U_x \times \mathrm{Prim}(\mathrm{End}_{\Gamma_0}(\beta)) \to S^*U_x$.

Proof. The first isomorphism is repeated from Lemma 3.5. The second one is obtained from the following:

(i) from the definition of $A_x = A_{U_x}$,
(ii) from the assumption that $\Gamma_x = \Gamma_0$,
(iii) from the fact that Γ_0 acts trivially on U_x, and
(iv) from the identifications

$$A_x^{\Gamma_0} \;:=\; \mathcal{C}_0(S^*U_x; \mathrm{End}(E))^{\Gamma_0} \simeq \mathcal{C}_0(S^*U_x) \otimes \mathrm{End}(\beta)^{\Gamma_0}.$$

The last isomorphism follows from Example 2.5 and the isomorphism $M_n(\mathbb{C}) \otimes A \simeq M_n(A)$, valid for any algebra A. The rest follows from Lemma 2.7.

Indeed, since both $\mathcal{C}_0(S^*U_x)$ and $\mathrm{End}(\beta)^{\Gamma_0}$ have only finite dimensional irreducible representations, we obtain $\mathrm{Prim}(A_x^{\Gamma_0}) = S^*U_x \times \mathrm{Prim}(\mathrm{End}_{\Gamma_0}(\beta)) \simeq S^*U_x \times \{1, 2, \ldots, N\}$, where we use the identification $\mathrm{Prim}(\mathcal{C}_0(S^*U_x)) \simeq S^*U_x$ and where the set $\{1, 2, \ldots, N\}$ is in natural bijection with the primitive ideal spectrum of the algebra $\mathrm{End}_{\Gamma_0}(\beta) \simeq \oplus_{j=1}^N M_{k_j}(\mathbb{C})$. The inclusion $\mathcal{C}_0(S^*U_x) = \mathcal{C}_0(S^*U_x)^{\Gamma_0} \to A_x^{\Gamma_0}$ is given by the unital inclusion $\mathbb{C} \to \oplus_{j=1}^{N} M_{k_j}(\mathbb{C})$. Hence the map $\operatorname{Prim}(A_x^{\Gamma_0}) \to S^*U_x$ identifies with the first projection in $S^*U_x \times \{1, 2, \ldots, N\} \to S^*U_x$. That is, it is a trivial covering, as claimed. □

The fibers of $\operatorname{Prim}(A_{M_0}^{\Gamma}) \to M_0/\Gamma$ are thus the simple factors of $\operatorname{End}(E_x)^{\Gamma_0}$, whose structure was determined in Example 2.5. We shall need the following remark similar to Remark 3.7, but simpler.

Remark 3.11. Let U be an open subset of a euclidean space, let V be a finite dimensional vector space and let V denote, by abuse of notation, also the trivial, vector bundle with fiber V. Then we have *natural* isomorphisms

$$\begin{aligned} \psi^{-1}(U; V) &\simeq \psi^{-1}(U) \otimes \operatorname{End}(V) \quad \text{and} \\ \psi^{0}(U; V) &\simeq \psi^{0}(U) \otimes \operatorname{End}(V)\,. \end{aligned}$$

Consequently, we also have the analogous isomorphisms for the completions

$$\begin{aligned} \overline{\psi^{-1}}(U; V) &\simeq \overline{\psi^{-1}}(U) \otimes \operatorname{End}(V) \quad \text{and} \\ \overline{\psi^{0}}(U; V) &\simeq \overline{\psi^{0}}(U) \otimes \operatorname{End}(V)\,. \end{aligned}$$

We are in position now to determine the kernel of $\mathcal{R}_{W_x}$, when x is in the principal orbit bundle. We will use the notation of Subsection 2.4 that was recalled at the beginning of this subsection as well as the notation of Subsection 2.2. In particular, recall that $\beta_j \in \widehat{\Gamma}_0$ and $\alpha \in \widehat{\Gamma}$ are said to be Γ_0-disjoint if β_j is *not* contained in the restriction of α to Γ_0. Also, Φ is the Frobenius isomorphism, Equations (13) and (14) and Corollary 3.10.

Proposition 3.12. *Let $\Gamma_x = \Gamma_0$, let $E_x = \beta = \oplus_{j=1}^{N} \beta_j^{k_j}$, and $\Phi : \mathcal{C}_0(S^*U_x) \otimes \operatorname{End}_{\Gamma_0}(\beta) \simeq A_x^{\Gamma_0} \to A_{W_x}^{\Gamma}$ be the Frobenius isomorphism of Corollary 3.10. Then*

(1) $\mathcal{C}_0(S^*U_x) \otimes \operatorname{End}_{\Gamma_0}(\beta_j^{k_j}) \subset \Phi^{-1}(\ker(\mathcal{R}_{W_x}))$ if β_j and α are Γ_0-disjoint, and

(2) $\mathcal{C}_0(S^*U_x) \otimes \operatorname{End}_{\Gamma_0}(\beta_j^{k_j}) \cap \Phi^{-1}(\ker(\mathcal{R}_{W_x})) = 0$ if β_j and α are Γ_0-associated.

In particular, Also, let $J \subset \{1, 2, \ldots, N\}$ be the set of indices j such that β_j and α are Γ_0-disjoint, then

$$\begin{aligned} \ker(\mathcal{R}_{W_x}) &= \Phi\big(\oplus_{j\in J}\, \mathcal{C}_0(S^*U_x) \otimes \operatorname{End}_{\Gamma_0}(\beta_j^{k_j}) \big) \quad \textit{and} \\ \pi_\alpha(B_M^{\Gamma})/\pi_\alpha(\overline{\psi^{-1}}(M; E)^{\Gamma}) &\simeq \Phi\big(\oplus_{j\notin J}\, \mathcal{C}_0(S^*U_x) \otimes \operatorname{End}_{\Gamma_0}(\beta_j^{k_j}) \big)\,. \end{aligned}$$

Proof. The proof is essentially a consequence of Proposition 2.4 by including U_x as a parameter, using also Lemma 3.8. To see how this is done, we will use the notation of that lemma, in particular, $W_x \simeq \Gamma \times_{\Gamma_0} U_x \simeq (\Gamma/\Gamma_0) \times U_x$ and $E \simeq \Gamma \times_{\Gamma_0} (U_x \times \beta)$. We identify W_x with $\Gamma \times_{\Gamma_x} U_x$, i.e. we work with $W_x = \Gamma \times_{\Gamma_x} U_x$.

Let π_α the fundamental morphism of restriction to the α-isotypical component, see Equations (1) and (11). Recall that $B_x := \overline{\psi^0}(U_x; E)$. Since Γ_x acts trivially on U_x, Remark 3.11 yields the Γ-equivariant isomorphisms

$$\operatorname{Ind}_{\Gamma_0}^{\Gamma}(B_x) \simeq \overline{\psi^0}(U_x) \otimes \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\operatorname{End}(\beta)) \subset B_{W_x}\,, \tag{33}$$

where the last inclusion is modulo the trivial identification given by $P \otimes f(s)(\gamma, x) = P(f(\gamma)s(\gamma))(x)$, $P \in \overline{\psi^0}(U_x)$, $f \in \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\operatorname{End}(\beta))$ and $s \in C_c(W_x, \operatorname{End}(E))$. Combining further Remark 3.11 with Remark 3.7, we further obtain the isomorphism

$$\overline{\psi^{-1}}(W_x; E) \simeq \overline{\psi^{-1}}(U_x) \otimes \operatorname{End}(\operatorname{Ind}_{\Gamma_0}^{\Gamma}(\beta))\,.$$

Lemma 3.8 and the exactness of the functor $V \to V^\Gamma$ give $\pi_\alpha(B_{W_x}^\Gamma) = \pi_\alpha \circ \Phi(B_x^{\Gamma_x}) + \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma)$. Hence we obtain $\pi_\alpha(B_{W_x}^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma) = \pi_\alpha \circ \Phi(B_x^{\Gamma_x})/\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) \cap \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma)$.

Let $\mathfrak{A}$ and $\mathfrak{J}$ be the image and, respectively, the kernel of $\pi_\alpha : \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\operatorname{End}(\beta))^\Gamma \to \operatorname{End}(p_\alpha \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\beta))$, which have been identified in Proposition 2.4 in terms of the set J. Recall next from Equation (23) that $L^2(W_x; E) = L^2(U_x) \otimes \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\beta)$, again Γ-equivariantly. Each time, the action is on the second component, since $\Gamma_0 = \Gamma_x$ acts trivially on $\overline{\psi^0}(U_x)$. The action of $\operatorname{Ind}_{\Gamma_0}^{\Gamma}(B_x) \subset B_{W_x}$ on $L^2(W_x; E) = L^2(U_x) \otimes \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\beta)$ is compatible with the tensor product decomposition of Equation (33), in the sense that $\overline{\psi^0}(U_x)$ acts on $L^2(U_x)$ and $\operatorname{Ind}_{\Gamma_0}^{\Gamma}(\operatorname{End}(\beta))$ acts on $\operatorname{Ind}_{\Gamma_0}^{\Gamma}(\beta)$. Also, $\operatorname{Ind}_{\Gamma_0}^{\Gamma}(B_x)^\Gamma \simeq \overline{\psi^0}(U_x) \otimes \operatorname{Ind}_{\Gamma_0}^{\Gamma}(\operatorname{End}(\beta))^\Gamma$, (we use this isomorphism to identify them). We obtain that

$$\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) = \pi_\alpha(\operatorname{Ind}_{\Gamma_0}^{\Gamma}(B_x)^\Gamma) = \overline{\psi^0}(U_x) \otimes \mathfrak{A}\,. \tag{34}$$

On the other hand, Corollary 2.9 then gives that $\pi_\alpha(\overline{\psi^{-1}}(W_x; \operatorname{End}(E))^\Gamma)$ is the algebra of Γ-invariant compact operators acting on the space $p_\alpha(L^2(W_x, \operatorname{End}(E)))$. Therefore, $\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A} \subset \pi_\alpha(\overline{\psi^{-1}}(W_x; \operatorname{End}(E))^\Gamma)$, since $\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A}$ consists of compact, Γ-invariant operators acting on $p_\alpha(L^2(W_x, E))$. Consequently,

$$\begin{aligned} \overline{\psi^{-1}}(U_x) \otimes \mathfrak{A} &\subset \pi_\alpha(\operatorname{Ind}_{\Gamma_0}^{\Gamma}(B_x)^\Gamma) \cap \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma) \\ &\subset \overline{\psi^0}(U_x) \otimes \mathfrak{A} \cap \mathcal{K}(p_\alpha L^2(W_x; E))^\Gamma \subset \overline{\psi^{-1}}(U_x) \otimes \mathfrak{A}\,, \end{aligned} \tag{35}$$

and hence we have equalities everywhere.

Recall from Corollary 3.10 that $A^{\Gamma}_{W_x} \simeq A_x^{\Gamma_x}$. We obtain that the map

$$\mathcal{R}_{W_x} : A^{\Gamma}_{W_x} \simeq B^{\Gamma}_{W_x}/\overline{\psi^{-1}}(W_x;E)^{\Gamma} \to \pi_\alpha(B^{\Gamma}_{W_x})/\pi_\alpha(\overline{\psi^{-1}}(W_x;E)^{\Gamma}) \tag{36}$$

becomes, up to the canonical isomorphisms above, the map

$$\begin{aligned} A_x^{\Gamma_x} \simeq \mathcal{C}_0(S^*U_x)\otimes \mathrm{End}_{\Gamma_0}(\beta) \ &\to \ \pi_\alpha(B^{\Gamma}_{W_x})/\pi_\alpha(\overline{\psi^{-1}}(W_x)^{\Gamma}) \\ &= \pi_\alpha\circ\Phi(B_x^{\Gamma_x})/\pi_\alpha\circ\Phi(B_x^{\Gamma_x})\cap \pi_\alpha(\overline{\psi^{-1}}(W_x)^{\Gamma}) \\ &\simeq \overline{\psi^{0}}(U_x)\otimes\mathfrak{A}/\overline{\psi^{-1}}(U_x)\otimes\mathfrak{A} \simeq \mathcal{C}_0(S^*U_x)\otimes\mathfrak{A}\,, \end{aligned} \tag{37}$$

with all maps being surjective and preserving the tensor product decompositions. This identifies the kernel of $\mathcal{R}_{W_x}$ with $\mathcal{C}_0(S^*U_x)\otimes\mathfrak{J}$ and the image of $\mathcal{R}_{W_x}$ with $\mathcal{C}_0(S^*U_x)\otimes\mathfrak{A}$. The rest of the statement follows from the identification of $\mathfrak{J}$ and $\mathfrak{A}$ in Proposition 2.4. □

Proposition 3.12 above and its proof give the following corollary.

Corollary 3.13. *We use the notation of Proposition 3.12 and we identify* $\mathrm{Prim}(\mathrm{End}(\beta))$ *with* $\{1,2,\ldots,N\}$ *as in Remark 2.5. Then the homeomorphism* $\mathrm{Prim}(A^{\Gamma}_{W_x}) \simeq S^*U_x\times\{1,2,\ldots,N\}$ *maps the set* $\Xi\cap\mathrm{Prim}(A^{\Gamma}_{W_x})$ *to* $S^*U_x\times J$. *In particular, the restriction* $\Xi\cap\mathrm{Prim}(A^{\Gamma}_{W_x})\to S^*U_x$ *of the central character is a covering as well.*

Proof. Using the notations of the proof of Proposition 3.12, we have that $\ker(\mathcal{R}_{W_x})$ has primitive ideal spectrum $S^*U_x\times\mathrm{Prim}(\mathfrak{J})$. We have $\Xi\cap\mathrm{Prim}(A^{\Gamma}_{W_x}) = S^*U_x\times\mathrm{Prim}(\mathfrak{A})$. □

The same methods yield the following result.

Corollary 3.14. *Let* $M_0 := M_{(\Gamma_0)}$, *the principal orbit bundle. The central character map* $\mathrm{Prim}(A^{\Gamma}_{M_0})\to S^*M_0/\Gamma$ *defined by the inclusion* $\mathcal{C}_0(S^*M_0/\Gamma)\subset Z(A^{\Gamma}_{M_0})$ *is a covering with typical fiber* $\mathrm{Prim}(\mathrm{End}(E_x)^{\Gamma_0})$ *such that* $\Xi\cap\mathrm{Prim}(A^{\Gamma}_{M_0})\to S^*M_0/\Gamma$ *is a subcovering, see* (27) *for the definition of* Ξ. *In particular,* $\Xi\cap\mathrm{Prim}(A^{\Gamma}_{M_0})$ *is open and closed in* $\mathrm{Prim}(A^{\Gamma}_{M_0})$.

Proof. The first statement is true locally, by Corollary 3.10, and hence it is true globally. Indeed, let $x\in M_0$, let $\xi\in S^*_xM_0$, and let $\rho\in\widehat{\Gamma}_x$ that appears in E_x (so $(\xi,\rho)\in\Omega_M$). We let $W_x\subset M_0\subset M$ be the typical tube with minimal isotropy $\Gamma_x=\Gamma_0$, as before. Let $Z_x := \mathcal{C}_0(S^*W_x)^{\Gamma}\subset Z_M = \mathcal{C}(S^*M)^{\Gamma}$. Then $\mathrm{Prim}(Z_xA^{\Gamma}_M)$ is an open neighborhood in $\mathrm{Prim}(A^{\Gamma}_{M_0})$ of the primitive ideal $\ker(\pi_{\xi,\rho})$, see Proposition 3.1 for notation and details. We have that $Z_xA^{\Gamma}_M = A^{\Gamma}_{W_x}$ and hence, on $\mathrm{Prim}(Z_xA^{\Gamma}_M)$, the central character is a covering, by Corollary 3.10. Similarly, its restriction to $\Xi\cap\mathrm{Prim}(Z_xA^{\Gamma}_M)$ is a covering by Corollary 3.13. □

Putting Corollary 3.14 and Proposition 3.12 together we obtain the following results.

Corollary 3.15. *Let M_0 be the principal orbit type of M. The ideal* $\ker(\mathcal{R}_{M_0}) = A^\Gamma_{M_0} \cap \ker(\mathcal{R}_M)$ *is defined by the closed subset* $\Xi_0 := \Xi \cap \mathrm{Prim}(A^\Gamma_{M_0})$ *of* $\mathrm{Prim}(A^\Gamma_{M_0})$ *consisting of the sheets of* $\mathrm{Prim}(A^\Gamma_{M_0}) \to S^*M_0/\Gamma$ *that correspond to the simple factors* $\mathrm{End}(E_{x\rho})^{\Gamma_0}$ *of* $\mathrm{End}(E_x)^{\Gamma_0}$ *with ρ and α Γ_0-associated.*

If Γ is abelian, then ρ and α are characters and saying that they are Γ_0-associated means, simply, that their restrictions to Γ_0 coincide: $\rho|_{\Gamma_0} = \alpha_{\Gamma_0}$. This is consistent with the definition given in [5].

3.5. **The non-principal orbit case.** As in the rest of the paper, we assume M to be connected. We will show in Theorem 3.17 that Ξ is the closure of Ξ_0 in $\mathrm{Prim}(A^\Gamma_M)$. To that end, we first construct a suitable basis of neighborhoods of $\mathrm{Prim}(A^\Gamma_M)$ using Lemma 3.2.

Remark 3.16. Let $\Gamma(\xi, \rho) \in \mathrm{Prim}(A^\Gamma_M)$, where we have used the description of $\mathrm{Prim}(A^\Gamma_M)$ provided in Proposition 3.1 as orbits of pairs $\xi \in S^*M$ and suitable $\rho \in \widehat{\Gamma}_\xi$. We construct a basis of neighborhoods $(V_{\xi,\rho,n})_{n\in\mathbb{N}}$ of $\Gamma(\xi, \rho)$ in $\mathrm{Prim}(A^\Gamma_M)$ as follows. Let $\xi \in S^*_x M$ (that is, ξ sits above $x \in M$) and we use the notation U_x and W_x of Equation (21), as always.

First, by choosing a different point ξ in its orbit, if necessary, we may assume that $\Gamma_0 \subset \Gamma_\xi$. Now let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a family of Γ_ξ-invariant neighborhoods of ξ in S^*U_x such that:

- for all n and $\gamma \in \Gamma \setminus \Gamma_\xi$, we have $\gamma\mathcal{O}_n \cap \mathcal{O}_n = \emptyset$,
- $\mathcal{O}_{n+1} \subset \mathcal{O}_n$ and $\bigcap_{n\in\mathbb{N}} \mathcal{O}_n = \{\xi\}$.

For any $n \in \mathbb{N}$, we choose a function $\varphi_n \in \mathcal{C}_c(\mathcal{O}_n)^{\Gamma_\xi}$ such that $\varphi_n \equiv 1$ on $\mathcal{O}_{n+1}$. Let $p_\rho \in \mathrm{End}(E_x)^{\Gamma_\xi}$ be the projection onto $E_{x\rho}$. We can assume the bundle E to be trivial on U_x and, using that, we first extend p_ρ constantly on $\mathcal{O}_n$ and then as an element $q_n \in \mathcal{C}_c(S^*U_x; \mathrm{End}(E_x))^{\Gamma_x}$ defined as

$$q_n := \begin{cases} \Phi_{\Gamma_\xi,\Gamma_x}(\varphi_n p_\rho) & \text{on } \Gamma_x\mathcal{O}_n \\ 0 & \text{on } S^*U_x \setminus \Gamma_x\mathcal{O}_n, \end{cases}$$

with $\Phi_{\Gamma_\xi,\Gamma_x}$ the Frobenius isomorphism of Equation (14). Let us set $\tilde{q}_n := \Phi_{\Gamma_x,\Gamma}(q_n) \in A^\Gamma_M$, where $\Phi_{\Gamma_x,\Gamma}$ is the Frobenius isomorphism of Equation (14). Finally, we associate to $\tilde{q}_n$ the open set

$$V_{\xi,\rho,n} := \{J \in \mathrm{Prim}(A^\Gamma_M) \mid \tilde{q}_n \notin J\}.$$

Recall from 3.2 that $V_{\xi,\rho,n}$ is an open subset of $\mathrm{Prim}(A_M^\Gamma)$. Moreover, it follows from our definition that $V_{\xi,\rho,n+1} \subset V_{\xi,\rho,n}$ and that $\bigcap_{n\in\mathbb{N}} V_{\xi,\rho,n} = \{\Gamma(\xi,\rho)\}$.

Recall that we are assuming that M is connected.

Theorem 3.17. *The closed set* $\Xi \subset \mathrm{Prim}(A_M^\Gamma)$ *defined by the ideal* $\ker(\mathcal{R}_M)$*: that is,* $\Xi := \mathrm{Prim}(A_M^\Gamma) \smallsetminus \mathrm{Prim}(\ker(\mathcal{R}_M))$*, is the closure in* $\mathrm{Prim}(A_M^\Gamma)$ *of the set* $\Xi_0 := \Xi \cap \mathrm{Prim}(A_{M_0}^\Gamma)$*, where* M_0 *is the principal orbit bundle of* M*.*

Proof. We have that $\overline{\Xi}_0 \subset \Xi$ since $\Xi_0 \subset \Xi$ and the latter is a closed set. Conversely, let $\mathfrak{P} \in \mathrm{Prim}(A_M^\Gamma) \setminus \overline{\Xi}_0$. We will show that $\mathfrak{P} \notin \Xi$. Let $\mathfrak{P}$ correspond to $(\xi,\rho) \in \Omega_M$, as in Proposition 3.1. We may assume that $\Gamma_0 \subset \Gamma_\xi$. Let x be projection of ξ onto M. Since the problem is local, we may also assume that $U_x \subset T_xM$, that $M = W_x := \Gamma \times_{\Gamma_x} U_x$, and that $E := \Gamma \times_{\Gamma_x} (U_x \times \beta)$ for some Γ_x-module β.

Using the notations of Remark 3.16, there exists $n > 0$ such that $V_{\xi,\rho,n} \cap \Xi_0 = \emptyset$. Let $\tilde{q}_n = \Phi_{\Gamma_x,\Gamma}(q_n)$ be the symbol defined in Remark 3.16. The description of Ξ_0 provided in Corollary 3.15, the definition of $V_{\xi,\rho,n}$, and the definition of $\tilde{q}_n$ imply that $\pi_{\zeta,\rho'}(\tilde{q}_n) = 0$ for any $\zeta \in S^*M_0$ and $\rho' \in \widehat{\Gamma}_0$ such that $\Gamma(\zeta,\rho') \in \Xi_0$, that is, such that ρ' and α are Γ_0-associated.

We next "quantize $\tilde{q}_n$" in an appropriate way, that is, we construct an operator $\widetilde{Q}_n \in B_{W_x}^\Gamma$ with symbol $\tilde{q}_n$ and with other convenient properties as follows. First, let $\chi \in \mathcal{C}_c^\infty(U_x)^{\Gamma_x}$ be such that $\chi\varphi_n = \varphi_n$, which is possible since φ_n has compact support. Then let $\psi \in \mathcal{C}^\infty(T_x^*M)^{\Gamma_x}$ be such that $\psi(0) = 0$ if $|\eta| < 1/2$ and $\psi(\eta) = 1$ whenever $|\eta| \geq 1$. Recall that in this proof $U_x \subset T_xM$ is identified with its image in $M = \Gamma \times_{\Gamma_x} U_x$ through the exponential map. Let for any symbol a

$$Op(a)f(y) := \int_{T_x^*M} \int_{U_x} e^{i(y-z)\cdot\eta}\, a(y,z,\eta) f(z) dz d\eta.$$

We shall use this for $a(y,z,\eta) := \chi(y)\psi(\eta)\tilde{q}_n\Big(\frac{\eta}{|\eta|}\Big)\chi(z)$, then set

$$Q := Op(a)\,, \quad \text{that is}$$

$$Qf(y) := \int_{T_x^*M} \int_{U_x} e^{i(y-z)\cdot\eta}\, \chi(y)\psi(\eta)\tilde{q}_n\left(\frac{\eta}{|\eta|}\right)\chi(z) f(z) dz d\eta$$

to be the standard pseudodifferential operator on U_x, associated to the symbol $a(y,z,\eta) := \chi(y)\psi(\eta)\tilde{q}_n\Big(\frac{\eta}{|\eta|}\Big)\chi(z)$. The operator Q is Γ_x-invariant by construction. Using the Frobenius isomorphism of Equation (14), we extend Q to the operator $\widetilde{Q}_n := \Phi(Q)$, which acts on

$M = W_x = \Gamma \times_{\Gamma_x} U_x$ (see also Equation (23) with regards to this isomorphism). Then $\widetilde{Q}_n \in \Psi^0(M;E)^\Gamma$, that is, it is Γ-invariant, by construction, and has principal symbol $\sigma_0(\widetilde{Q}_n) = \tilde{q}_n$.

Now let $x_0 \in M_0 \cap U_x$, where, we recall, $M_0 := M_{(\Gamma_0)}$ denotes the principal orbit bundle. We have

$$L^2(W_{x_0};E) = \mathrm{Ind}_{\Gamma_0}^{\Gamma}\big(L^2(U_{x_0};\beta)\big) = L^2\big(U_{x_0};\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)\big),$$

where $\beta = E_{x_0} = E_x$ by the assumption that $E := \Gamma \times_{\Gamma_x} (U_x \times \beta)$.

Let $\beta_j \in \widehat{\Gamma_0}$ be the isomorphism classes of the Γ_ξ-submodules of β and $k_j \geq 0$ is the dimension of the corresponding β_j-isotypical component in β, so that $\beta \simeq \oplus_{j=1}^N \beta_j^{k_j}$, as Γ_0-modules, as before. Thus

$$L^2(W_{x_0};E) \simeq \bigoplus_{j=1}^N L^2(U_{x_0};\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j}))\,.$$

Recall that the α-isotypical component of $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j})$ is given by $\alpha \otimes \mathrm{Hom}_\Gamma(\alpha, \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j}))$, which is non-zero if, and only if, α and β_j are Γ_0-associated, by the Frobenius isomorphism. Hence, passing to the α-isotypical components, we have

$$L^2(W_{x_0};E)_\alpha := p_\alpha L^2(W_{x_0};E) = \bigoplus_{j\in J^c} L^2(U_{x_0};\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j}))_\alpha\,,$$

where $J \subset \{1,\dots,N\}$ is the set of indices such that $\beta_j \in \widehat{\Gamma}_0$ and α are Γ_0-disjoint; J^c is its complement (i.e. $\beta_j \in \widehat{\Gamma}_0$ and α are Γ_0-associated).

Let $p_J \in \mathrm{End}(\beta)^{\Gamma_0}$ be the projector onto $\bigoplus_{j\in J^c} \beta_j^{k_j}$. Recall that $\pi_{\zeta,\beta_j}(\tilde{q}_n) = 0$ for any $(\zeta,\beta_j) \in S^*M_0 \times \widehat{\Gamma}_0$ with $j \notin J$. Therefore $\tilde{q}_n(\zeta)p_J = 0$, for all $\zeta \in S^*M_0$. Since S^*M_0 is dense in S^*M, this implies that $\tilde{q}_n p_J = 0$. Thus

$$\widetilde{Q}_n p_J = Op(\chi\psi\tilde{q}_n\chi)p_J = Op(\chi\psi\tilde{q}_n\chi p_J) = 0\,.$$

Hence for any $f \in L^2(W_{x_0};E)_\alpha$, we have that $\widetilde{Q}_n f = 0$. This is true for any $x_0 \in M_0$, so we conclude that $\widetilde{Q}_n$ is zero on $L^2(M_0;E)_\alpha$. Since M_0 has measure zero complement in M, we have $L^2(M_0;E)_\alpha = L^2(M;E)_\alpha$; therefore $\pi_\alpha(\widetilde{Q}) = 0$. This implies that $\mathcal{R}_M(\widetilde{q}) = 0$, while $\pi_{\xi,\rho}(\widetilde{q}) = 1$. Thus $\Gamma(\xi,\rho) \notin \Xi$, which concludes the proof. □

Our question now is to decide whether some given $\Gamma(\xi,\rho)$ is in Ξ or not. Recall that ρ and α are said to be Γ_0-associated if $\mathrm{Hom}_{\Gamma_0}(\rho,\alpha) \neq 0$. The set $X_{M,\Gamma}^\alpha$ was defined in the introduction as the set of pairs $(\xi,\rho) \in T^*M \setminus \{0\} \times \widehat{\Gamma}_\xi$ for which there is an element $g \in \Gamma$ such that $g\cdot\rho$ and α are Γ_0-associated.

Proposition 3.18. *We use the notation in the last two paragraphs. We have $\Gamma(\xi,\rho) \in \Xi$ if, and only if, there is a $g \in \Gamma$ such that $g \cdot \rho$ and α are Γ_0-associated. In other words, we have that $\Xi \simeq X^{\alpha}_{M,\Gamma}/\Gamma$.*

Proof. Let $\Gamma(\xi,\rho) \in \mathrm{Prim}(A^{\Gamma}_M)$, with $x \in M$ the base point of ξ. We can assume (by choosing a different element in the orbit if needed) that $\Gamma_0 \subset \Gamma_\xi$. Let $\tilde{q}_n \in A^{\Gamma}_M$ be the element defined in Remark 3.16 and $V_{\xi,\rho,n}$ the corresponding neighbourhood of $\Gamma(\xi,\rho)$ in $\mathrm{Prim}(A^{\Gamma}_M)$.

There is a Γ_x-equivariant isomorphism $E|_{U_x} \simeq U_x \times \beta$, where $\beta = E_x$ is a Γ_x-module. Since $\Gamma_0 \subset \Gamma_x$, we may decompose β into Γ_0-isotypical components, i.e. $\beta = \bigoplus_{j=1}^{N} \beta_j^{k_j}$, with the usual notation. If $\eta \in \mathcal{O}_n$, then $\pi_{\eta,\beta_j}(\tilde{q}_n) = \varphi_n(\eta)\pi_{\beta_j}(p_\rho)$. Therefore, for any $\eta \in S^*M$, we have

$$\pi_{\eta,\beta_j}(\tilde{q}_n) = 0 \Leftrightarrow \mathrm{Hom}_{\Gamma_0}(\beta_j,\rho) = 0 \text{ or } \tilde{q}_n(\eta) = 0.$$

This implies that

$$V_{\xi,\rho,n} \cap \Xi_0 = \{\Gamma(\eta,\beta) \in \Xi_0 \mid \tilde{q}_n(\eta) \neq 0 \text{ and } \mathrm{Hom}_{\Gamma_0}(\beta,\rho) \neq 0\}$$

It follows from the determination of Ξ_0 in Corollary 3.15 that $V_{\xi,\rho,n} \cap \Xi_0 \neq \emptyset$ if, and only if, we have $\mathrm{Hom}_{\Gamma_0}(\rho,\alpha) \neq 0$. Now $\Xi = \overline{\Xi_0}$ by Theorem 3.17. Since the open sets $(V_{\xi,\rho,n})_{n\in\mathbb{N}}$ form a basis of neighborhoods of $\Gamma(\xi,\rho)$, we conclude that $\Gamma(\xi,\rho) \in \Xi$ if, and only if, we have $\mathrm{Hom}_{\Gamma_0}(\rho,\alpha) \neq 0$. □

We obtain the following corollary.

Corollary 3.19. *Let us assume that Γ acts freely on a dense open subset of M. Then $\Xi = \mathrm{Prim}(A^{\Gamma}_M)$.*

Proof. The assumption on the action implies that $\Gamma_0 = \{1\}$. If $\xi \in T^*M \setminus \{0\}$ and $\rho \in \widehat{\Gamma}_\xi$, then ρ and α are always $\{1\}$-associated. The Corollary then follows from Proposition 3.18. □

4. Applications and extensions

We now prove the main result of the paper on the characterization of Fredholm operators and discuss some extensions of our results.

4.1. **Fredholm conditions.** We now turn to the proof of our main result. We assume that M is a compact smooth manifold. We have the following Γ–equivariant version of Atkinson's theorem.

Proposition 4.1. *Let V be a unitary Γ–module and $P \in \mathcal{L}(V)^{\Gamma}$ be a Γ–equivariant bounded operator on V. We have that P is Fredholm if, and only if, it is invertible modulo $\mathcal{K}(V)^{\Gamma}$, in which case, we can choose the parametrix (i.e. the inverse modulo the compacts) to also be Γ-invariant.*

Proof. See for example [5, Proposition 5.1]. □

Corollary 4.2. *Let* $P \in \overline{\psi^0}(M;E)^\Gamma$ *and* $\alpha \in \hat{\Gamma}$. *We have that* $\pi_\alpha(P)$ *is Fredholm on* $L^2(M;E)_\alpha$ *if, and only if,* $\pi_\alpha(P)$ *is invertible modulo* $\pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma)$ *in* $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$.

We are now in a position to prove the main result of this paper, Theorem 1.2.

Proof ot Theorem 1.2. As in [5, Section 2.6], we may assume that $P \in \overline{\psi^0}(M;E)^\Gamma$. Corollary 4.2 then states that $\pi_\alpha(P)$ is Fredholm if, and only if, the image of its symbol $\sigma(P)$ is invertible in the quotient algebra

$$\mathcal{R}_M(A_M^\Gamma) = \pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma).$$

The isomorphism $\mathrm{Prim}(A_M^\Gamma) \simeq \Omega_M/\Gamma$ of Proposition 3.1 and Proposition 3.18 identify the primitive ideal spectrum Ξ of $\mathcal{R}_M(A_M^\Gamma)$ with the set $X_{M,\Gamma}^\alpha/\Gamma$. Therefore $\mathcal{R}_M(\sigma(P))$ is invertible if, and only if, the endomorphism $\pi_{\xi,\rho}(\sigma(P))$ is invertible for all $(\xi,\rho) \in X_{M,\Gamma}^\alpha$, i.e. if, and only if, P is α-elliptic. □

4.2. **The abelian case** [5]**.** Many statements and definitions become easier in the case of abelian groups. This is true in particular for the notion of Γ_0-associated representations using the following observation.

If Γ_i, $i = 1, 2$, are both abelian, then the irreducible representations α_i are characters, that is, morphisms $\alpha_i : \Gamma_i \to \mathbb{C}^*$, and we have that they are H-associated for some subgroup H if, and only if, $\alpha_1|_H = \alpha_2|_H$.

Let α be an irreducible representation of Γ. When Γ is abelian, the conjugacy class of isotropy subgroups corresponding to the principal orbit type of the action has only one element, namely Γ_0. In that case, the set $X_{M,\Gamma}^\alpha$ defined in Equation (7) of the introduction has the simpler expression:

$$X_{M,\Gamma}^\alpha = \{(\xi, \alpha|_{\Gamma_0}) \mid \xi \in T^*M \setminus \{0\}\}.$$

As a consequence, it is easier to check the α-ellipticity for an operator P in the abelian case. Let E, F be Γ-equivariant vector bundles over M and set $\alpha_0 := \alpha|_{\Gamma_0}$. Recall that, for any $x \in M$, we denote by $E_{x\alpha_0}$ the α_0-isotypical component of E_x, seen as a Γ_0-representation. We then recover the main result of [5]. Indeed, Theorem 1.2 can then be stated as follows:

Theorem 4.3. [5, Theorem 1.2] *Let* Γ *be a finite,* abelian *group acting on a smooth, compact manifold* M *and let* $P \in \psi^m(M;E,F)^\Gamma$. *Then, for any* $s \in \mathbb{R}$, *the following are equivalent:*

(1) *the operator* $\pi_\alpha(P) : H^s(M;E)_\alpha \to H^{s-m}(M;F)_\alpha$ *is Fredholm,*

(2) for all $(x,\xi) \in T^*M \setminus \{0\}$, the restriction of $\sigma(P)(x,\xi)$ defines an isomorphism

$$\pi_{\alpha_0}(\sigma(P)(x,\xi)) : E_{x\alpha_0} \to F_{x\alpha_0} .$$

4.3. **Particular cases.** In this section, Γ denote a finite group and $\hat{1}$ denotes the trivial representation. We denote again by M a Γ-manifold.

4.3.1. *Scalar operators.* Our main theorem becomes quite explicit when we are dealing with scalar operators, i.e. when the vector bundles $E_i = M \times \mathbb{C}$, where $\mathbb{C}$ denotes the trivial representation of Γ.

Proposition 4.4. *Let $P : H^s(M) \to H^{s-m}(M)$ be a Γ-invariant pseudodifferential operator. Let $\alpha \in \widehat{\Gamma}$. Then P is α-elliptic if and only if $\sigma(P)(\xi)$ is invertible for all $\xi \in T^*M \setminus \{0\}$ such that α is Γ_0-associated to $\hat{1}_{\Gamma_\xi}$.*

Proof. Let $(\xi,\rho) \in X^\alpha_{M,\Gamma}$. If $\rho \neq \hat{1}_{\Gamma_\xi}$ then $\mathbb{C}_\rho = 0$ and then $\pi_\rho(\sigma(P)(\xi)) : 0 \to 0$ is invertible. Now if $\rho = \hat{1}_{\Gamma_\xi}$ then $(\xi,\rho) \in X^\alpha_{M,\Gamma}$ if and only if α is Γ_0-associated to $\hat{1}_{\Gamma_\xi}$. □

4.3.2. *Trivial action.* Assume that M is connected and Γ acts trivially on M. Our assumption implies that $\Gamma_0 = \Gamma_\xi = \Gamma$, for all $\xi \in T^*M \setminus \{0\}$. It follows that $\rho \in \widehat{\Gamma}_\xi$ is Γ_0-associated to $\alpha \in \widehat{\Gamma}$ if and only if $\alpha = \rho$.

Let $E \to M$ be a Γ-equivariant vector bundle. For any $x \in M$, recall that we denote $E_{x\alpha}$ the α-isotypical component of E_x. Assuming M to be connected, we have that $E_\alpha = \bigcup_{x \in M} E_{x\alpha}$ form a Γ-equivariant sub-vector bundle of E.

Proposition 4.5. *Assume that Γ acts trivially on M and let $\alpha \in \widehat{\Gamma}$. Let E, F be two Γ-equivariant vector bundles over M and let $P \in \psi^m(M; E, F)^\Gamma$. Then for any $s \in \mathbb{R}$, the following are equivalent*

(1) $\pi_\alpha(P) : H^s(M; E_\alpha) \to H^{s-m}(M; F_\alpha)$ is Fredholm,

(2) for all $(x,\xi) \in T^*M \setminus \{0\}$, the morphism

$$\pi_\alpha(\sigma(P)(x,\xi)) : E_{x\alpha} \to F_{x\alpha}$$

is invertible,

(3) for all $(x,\xi) \in T^*M \setminus \{0\}$, the morphism

$$\sigma_m(P) \otimes \mathrm{id}_{\alpha^*}(x,\xi) : \mathrm{Hom}_\Gamma(\alpha, E_x) \to \mathrm{Hom}_\Gamma(\alpha, F_x)$$

is invertible.

Note that we recover the usual Fredholmness result for the elliptic operator $p_{F_\alpha} P p_{E_\alpha} \in \psi^m(M; E_\alpha, F_\alpha)$.

Proof. The equivalence between (1) and (2) is a direct consequence of Theorem 1.2.

Let us check the equivalence of (1) and (3). First note that

$$(H^s(M,E)\otimes\alpha)^\Gamma = H^s(M,(E\otimes\alpha)^\Gamma),$$

since the action of Γ on M is trivial. The operator $\pi_\alpha(P)$ is Fredholm if, and only if, the following pseudodifferential operator $P_\alpha : H^s(M,\mathrm{Hom}(\alpha,E)^\Gamma) \to H^{s-m}(M,\mathrm{Hom}(\alpha,F)^\Gamma)$ defined for any $v^*\in\alpha^*$ and $s\in\mathcal{C}^\infty(M,E)$ by $P_\alpha(v^*s) = v^*Ps$ is Fredholm. Furthermore, the operator P_α is Fredholm if, and only if, it is elliptic, that is if, and only if, $\sigma_m(P)\otimes\mathrm{id}_{\alpha^*}(x,\xi) : \mathrm{Hom}_\Gamma(\alpha,E_x)\to\mathrm{Hom}_\Gamma(\alpha,F_x)$ is invertible for any $(x,\xi)\in T^*M\setminus\{0\}$. Note that the invertibility of $\sigma_m(P)\otimes\mathrm{id}_{\alpha^*}(x,\xi)$ is equivalent to the invertibility of $\pi_\alpha(\sigma_m(P)(x,\xi))$ by definition, so this is consistent with (2). ☐

4.3.3. *Free action on a dense subset.* As in the previous sections, the group Γ is finite and acts continuously on the manifold M. We consider vector bundles $E,F\to M$.

Proposition 4.6. *Assume that Γ acts freely on a dense subset in M, and let $P\in\psi^m(M;E,F)^\Gamma$. For any $\alpha\in\widehat{\Gamma}$, we have that P is α-elliptic if, and only if, P is elliptic.*

Proof. It follows from Corollary 3.19 that $X^\alpha_{M,\Gamma} = X_{M,\Gamma}$. Thus the operator P_α is α-elliptic if, and only if, the sum $\bigoplus_{\rho\in\widehat{\Gamma}_\xi}\pi_\rho(\sigma_m(P)(\xi)) = \sigma_m(P)(\xi)$ is invertible for all $\xi\in T^*M\setminus\{0\}$, i.e. if, and only if, P is elliptic. ☐

4.4. **Simonenko's localization principle.** In this section we give a characterization using Simonenko's principle [43] for $\pi_\alpha(P)$ to be Fredholm. We start with two preliminary lemmas that we use to obtain a proof of Simonenko's principle in the classical case and in our case.

We assume as before that M is a compact smooth manifold endowed with an action of a compact group Γ.

4.4.1. *Simonenko's general principle.* Let A be a unital C^*-algebra and $Z=\mathcal{C}(\Omega_Z)$ unital sub-C^*-algebra in A, i.e. $1_Z = 1_A$. An element $a\in A$ is said to *have the strong Simonenko local type property* with respect to Z if, for every $\phi,\psi\in Z$ with compact disjoint supports, $\phi a\psi = 0$.

Lemma 4.7. *The set $B\subset A$ of elements a satisfying the strong Simonenko local property is the set of elements of A commuting with Z.*

Proof. We are going to show that the set of elements $a\in A$ with the strong Simonenko local type property is a C^*-algebra B containing Z

and that every irreducible representation of B restricts to a scalar on Z, and hence that Z commutes with B.

Let us show that B is a sub-C^*-algebra of A. Note that B is not empty since $Z \subset B$. To show that B is a sub-C^*-algebra, the only fact that is not trivial is that $ab \in B$, for all $a, b \in B$. Let ϕ and $\psi \in Z$ with disjoint compact supports and let θ be a function equal to 1 on $\mathrm{supp}(\psi)$ and 0 on $\mathrm{supp}(\phi)$ using Urysohn lemma. Then we have

$$\phi ab\psi = \phi a(\theta + 1 - \theta)b\psi = \phi a\theta b\psi + \phi a(1-\theta)b\psi = 0, \tag{38}$$

since $\phi a\theta = 0$ and $(1-\theta)b\psi = 0$.

Let $\pi : B \to \mathcal{L}(H)$ be an irreducible representation of B. First, let us show that for any $\phi, \psi \in Z$ with disjoint support, we either have $\pi(\phi) = 0$ or $\pi(\psi) = 0$. Indeed we have $\pi(\phi)\pi(a)\pi(\psi) = 0$ since $\phi a\psi = 0$, for any $a \in B$. Assume that $\pi(\psi) \neq 0$ then there is $\eta \in H$ such that $\pi(\psi)\eta \neq 0$. Now, π is irreducible so we get that the set $\{\pi(a)\pi(\psi)\eta,\ a \in B\}$ is dense in H. Thus $\pi(\phi) = 0$ on a dense subspace of H and so on H.

Assume now that $\pi(Z) \neq \mathbb{C}1_H$. Then there exist two distinct characters $\chi_0, \chi_1 \in \mathrm{Sp}(\pi(Z))$. Denote by $h_\pi : \mathrm{Sp}(\pi(Z)) \to \mathrm{Sp}(Z)$ the injective map adjoint to π, and choose $\phi, \psi \in \mathcal{C}(\mathrm{Sp}(Z))$ with disjoint supports such that $\phi(h_\pi(\chi_0)) = 1$ and $\psi(h_\pi(\chi_1)) = 1$. Then $\pi(\phi)(\chi_0) = 1$ and $\pi(\psi)(\chi_1) = 1$, which contradicts the fact that either $\pi(\phi) = 0$ or $\pi(\psi) = 0$.

□

Lemma 4.8. *Let A be a unital C^*-algebra and $Z = \mathcal{C}(\Omega_Z)$ be a unital sub-C^*-algebra as before. Then for every primitive ideal $J \subset A$, there exists $\omega \in \Omega_Z$ such that $\omega A \subset J$.*

Recall that a family $(\varphi_i)_{i \in I}$ of morphisms of a C^*-algebra A is said to be *exhaustive* if any primitive ideal contains $\ker \varphi_i$, for some $i \in I$ [38]. Then Lemma 4.8 can also be formulated by saying that the family of morphisms $\chi_\omega : A \to A/\omega A$, for $\omega \in \Omega_Z$, is exhaustive for A.

Proof. Let $J = \ker \pi \subset A$ be a primitive ideal. Since A is a sub-algebra commuting with Z we get that $\pi(Z) \subset \pi(A)'$ so Schur's Lemma implies that $\pi(Z) \subset \mathbb{C}id$. We deduce that $\pi_{|Z}$ is irreducible, and $\pi|_Z \neq 0$ since $1_Z = 1_A$. It follows that $\ker(\pi_{|Z})A \subset J$. □

Definition 4.9. Denote by $\mathcal{H} = L^2(M)$. An operator $P \in \mathcal{L}(\mathcal{H})$ is said to be *locally invertible* at $x \in M$ if there are a neighbourhood V_x of x and operators Q_1^x and $Q_2^x \in \mathcal{L}(\mathcal{H})$ such that

$$Q_1^x P\phi = \phi \quad \text{and} \quad \phi P Q_2^x = \phi, \qquad \text{for any } \phi \in \mathcal{C}_c(V_x). \tag{39}$$

The operator P is said to be *locally invertible* if it is locally invertible at any $x \in M$.

Let $\Psi_M \subset \mathcal{L}(\mathcal{H})$ be the C^*-algebra of all $P \in \mathcal{L}(\mathcal{H})$ such that $\phi P \psi \in \mathcal{K}(\mathcal{H})$, for all $\phi, \psi \in \mathcal{C}(M)$ with disjoint support. We denote by $\mathcal{B}_M$ the image of Ψ_M in the Calkin algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. We know by Lemma 4.7 that

$$\mathcal{B}_M = \{P \in \mathcal{Q}(H) \mid \phi P = P\phi \text{ for all } \phi \in \mathcal{C}(M)\}.$$

Proposition 4.10 (Simonenko's principle)**.** *If $P \in \Psi_M$, then P is locally invertible if, and only if, it is Fredholm.*

Proposition 4.10 is a particular case of the equivariant version that we prove in Proposition 4.11; we thus refer to Proposition 4.11 for the proof.

4.4.2. *Application.* We have always said that Γ is finite, so let us denote now the group by G and assume that G is compact. Let us return to the Fredholm condition for $\pi_\alpha(P)$. Denote by $\mathcal{H} := L^2(M, E)$ and by $\mathcal{H}_\alpha$ the α-isotypical component associated to $\alpha \in \hat{G}$.

We say that $P \in \mathcal{L}(\mathcal{H})$ is locally α-invertible at $x \in M$ if there are a G-invariant neighbourhood V_x of Γx and operators Q_1^x and $Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha)$ such that

$$Q_1^x \pi_\alpha(P)\phi = \phi \qquad \text{and} \qquad \phi\pi_\alpha(P)Q_2^x = \phi,$$

for any $\phi \in \mathcal{C}(M)^G$ supported in V_x.

We denote by Ψ_M^G the G-invariant elements in the C^*-algebra Ψ_M, which was defined in the previous subsection.

Proposition 4.11 (Simonenko's equivariant principle)**.** *Let $P \in \Psi_M^G$. Then P is locally α-invertible if and only if $\pi_\alpha(P)$ is Fredholm.*

Proof. Let $\mathcal{B}_M^\alpha$ be the image of Ψ_M^G in the Calkin algebra $\mathcal{Q}(\mathcal{H}_\alpha)$. We know from Lemma 4.7 that

$$\mathcal{B}_M^\alpha = \{P \in \mathcal{Q}(\mathcal{H}_\alpha) \mid \phi P = P\phi,\ \forall \phi \in \mathcal{C}(M)^G\}.$$

Assume that P is locally α-invertible, i.e. $\forall x \in M$, there are a neighborhood V_x of Gx and operators $Q_1^x, Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha)$ such that $Q_1^x\pi_\alpha(P)\phi = \phi$ and $\phi\pi_\alpha(P)Q_2^x = \phi$, for any $\phi \in \mathcal{C}(M)^G$ supported in V_x. Denote by $\chi_x : \mathcal{B}_G^\alpha \to \mathcal{B}_G^\alpha / Gx\mathcal{B}_G^\alpha$. We use the same notation for $\pi_\alpha(P)$ and its projection in $\mathcal{Q}(\mathcal{H}_\alpha)$. We have that

$$\chi_x(Q_1^x\pi_\alpha(P)\phi) = \chi_x(Q_1^x)\chi_x(\pi_\alpha(P))\chi_x(\phi) = \chi_x(\phi).$$

Since $\chi_x(\phi) = 1$, we get:

$$\chi_x(Q_1^x)\chi_x(\pi_\alpha(P)) = 1.$$

And similarly,

$$\chi_x(\pi_\alpha(P))\chi_x(Q_2^x) = 1.$$

Then by Lemma 4.8, we obtain that $\pi_\alpha(P)$ is invertible in $\mathcal{B}_M^\alpha$ and so it is Fredholm.

Now assume that $\pi_\alpha(P)$ is Fredholm and let Q be an inverse modulo $\mathcal{K}(\mathcal{H}_\alpha)$ for $\pi_\alpha(P)$, i.e. $\pi_\alpha(P)Q = id + K$ and $Q\pi_\alpha(P) = id + K'$, with $K, K' \in \mathcal{K}(\mathcal{H}_\alpha)$. Using Proposition 4.1 and Lemma 2.9, we can assume that $K = \pi_\alpha(k)$ and $K' = \pi_\alpha(k') \in \mathcal{K}(\mathcal{H}_\alpha) = \pi_\alpha(\mathcal{K}(\mathcal{H})^G)$. Let $\chi \in \mathcal{C}(M)^G$ be equal to 1 on a G-invariant neighbourhood V_x of Gx and let $\phi \in \mathcal{C}(M)^G$ be supported in V_x then

$$\phi\chi\pi_\alpha(P)Q\chi = \phi\chi^2 + \phi\chi K\chi \qquad \text{and} \qquad \chi\pi_\alpha(P)Q\chi\phi = \chi^2\phi + \chi K'\chi\phi.$$

Since ϕ is supported in V_x, we have $\phi\chi = \phi$ and so

$$\phi\pi_\alpha(P)Q\chi = \phi(1 + \chi K\chi) \qquad \text{and} \qquad \pi_\alpha(P)Q\chi\phi = (1 + \chi K'\chi)\phi.$$

Choosing V_x small enough, we may assume that $\|\chi K\chi\| < 1$ and $\|\chi K'\chi\| < 1$. It follows that $(1 + \chi K\chi)$ and $(1 + \chi K'\chi)$ are invertible and this implies

$$\phi\pi_\alpha(P)\big(Q\chi(1 + \chi K\chi)^{-1}\big) = \phi \quad \text{and} \quad \big((1 + \chi K'\chi)^{-1}\chi Q\big)\pi_\alpha(P)\phi = \phi,$$

i.e. P is locally α-invertible. □

We now recall the definition of G-transversally elliptic operator, see [2]. Assume that M is a compact smooth manifold and that G is a compact Lie group acting on M. Denote by $\mathfrak{g}$ the Lie algebra of G. Recall that any $X \in \mathfrak{g}$ defines as usual the vector field X_M^* given by $X_M^*(m) = \frac{d}{dt}_{|t=0} e^{tX} \cdot m$. Denote by $\pi : T^*M \to M$ the projection and let us introduce accordingly with [2] the G-transversal space

$$T_G^*M := \{\alpha \in T^*M \mid \alpha(X_M^*(\pi(\alpha))) = 0, \forall X \in \mathfrak{g}\}.$$

A G-invariant classical pseudodifferential operator P of order m is said *G-transversally elliptic* if its principal symbol is invertible on $T_G^*M \setminus \{0\}$.

Corollary 4.12. *Assume that M is compact, G is a compact Lie group and let $P \in \psi^0(M; E)^G$ be a G-transversally elliptic operator. Then P is locally α-invertible for any $\alpha \in \widehat{G}$.*

Corollary 4.13. *Assume that M is compact and Γ is finite. Let $P \in \psi(M; E, F)^\Gamma$ and $\alpha \in \widehat{\Gamma}$. Then the following are equivalent:*

(1) $P : H^s(M; E)_\alpha \to H^{s-m}(M; F)_\alpha$ is Fredholm for any $s \in \mathbb{R}$,
(2) P is α-elliptic,
(3) P is locally α-invertible.

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